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A TREATISE

ON THE

THEORY

OF

ALGEBRAICAL EQUATIONS.

BY

THE REV. ROBERT MURPHY, M.A.,

&c. &c.

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PREFACE.

THE commencement of the publication of a "Treatise on the Theory of Algebraical Expressions," of which the first number appeared in 1831, under the direction of the "Society for the Diffusion of Useful Knowledge," having been accidentally interrupted, I was directed by the same Society to compose the present Treatise, keeping in sight the views of the author of the "Algebraical Expressions," at the same time keeping pace with the advancement made, since the former date, on this subject.

The main view of the author of the work quoted, I learned, was to conduct his subject so as insensibly to lead the learner from pure algebraical theories to a knowledge of the principles on which the more advanced branches of analysis depend. To this advice from an excellent analyst I have adhered as well as I was able; but, in consideration of the recent progress of the "Theory of Equations," I felt it necessary to alter the plan, assuming however the propositions proved in the other work, to which therefore the reader will find several subsequent references.

I will now make a short statement of the plan adopted in the present work, premising that no treatise with exclusively the same object has been published of late, as far as I know, either at home or abroad. To collect and methodically digest the scattered elements of this theory, as far as its present advanced state imports, was attended with no inconsiderable difficulties; therefore, though an object of great utility has been, I hope, obtained by the composition of this work, it cannot be expected to be altogether faultless.

Before examining algebraical equations theoretically, it appeared necessary to convey a precise idea of the continuous nature of algebraic functions, and to show that their numerical magnitudes may be extended through every quantity from negative to positive infinity, notwithstanding the existence of certain maxima and minima values. This subject is discussed in a series of propositions, the more clearly to impress the reader with the steps of the reasoning. Having attained this object, the ordinary properties of equations relative to the existence, number, limits, and symmetrical relations of the roots, followed as easy consequences; on these deduced properties I have not much dilated, as they have been already ably treated in the work before referred to.

I have then given the theorems of both Sturm and Fourier relative to the discovery of the number of real and imaginary roots of an equation, the combination of which with the methods of approximation due to Newton and Lagrange conducts to the solution of all numerical equations of finite dimensions, except for imaginary roots, for the discovery of which I have employed a method deduced from recurring series. These numerical applications I have illustrated by examples in a later part of the work.

The formation of literal equations being understood, I have explained the logarithmic method for obtaining with rapidity the series which analytically represent the different roots and their functions; and have then shown how to effect some general and useful transformations of equations, and explained the algebraical solutions of the equations of inferior degrees, and the analytical meaning of the different surd parts which constitute the roots.

The theory of binomial equations is treated much in the manner of Lagrange, and the methods for the general resolution of equations are then discussed; and wherever useful applications to the kindred branches arose, I have supplied them in the form of *Scholia*, in order to preserve a proper arrangement of the subject more especially treated.

After giving several useful analytical results springing from the employment of the methods before given, I have passed on to discuss recurring series, which have been used from an early date for the solution of equations. I have then pointed out the useful extension made but left unproved by Fourier; I have supplied the proofs for that part which was correct, and substituted right theorems for those in which he has committed errors.

After then giving the various methods of approximation to the roots, and using all the appliances by which they may mutually assist each other, and thereby facilitate the numerical solution—on which occasion I have also considered several properties of continued fractions—I have, in conclusion, considered the properties of general classes of equations of finite and infinite dimensions, and shown in what cases the theory of the former may or may not be applicable to the latter.

All parts of the work I have taken care to illustrate with numerical or more general examples, and to draw such inferences as will be found useful in the higher branches of analysis. A glance over the table of contents, which I hope will form a useful epitome of the whole subject, will convey a more complete idea of the nature and extent of the matters here treated.

I have availed myself of an accidental delay in the publication of this Treatise, to revise, correct, and augment different parts, with the view of rendering the work as complete as possible.

R. M.

February 3, 1838.

ANALYTICAL TABLE

OF MATTERS CONTAINED IN THIS TREATISE ON ALGEBRAICAL EQUATIONS.

THE THEORY OF EQUATIONS.

Article 1. The direct processes of arithmetic were at first purely computative, as addition and multiplication. An expertness in performing these operations, arising from habitude and the assistance of tables committed to memory, easily conducted to the inverse processes of subtraction and division, and to the compound process of proportion. When the direct process, however, became more complex, as when it consisted of various multiplications and additions, the discovery of methods for finding the unknown quantity or quantities was a matter of extreme difficulty to arithmeticians, and but for the employment of *general symbols* denoting the quantities on which the operations were performed, the discovery of the proper inverse processes of arithmetic would have progressed very slowly.

These symbols however were employed: the problems on which they were engaged conducted generally to simple equations, the results of which furnished *rules* for Single and Double False Position, Alligation, and other purposes of arithmetic to be found in the old treatises on this subject.

Questions, in which the quantity sought was multiplied by itself, or which in symbols conducted to quadratic equations, were afterwards attempted by the rule of double false position, which necessarily gave only an approximation. For greater readiness in obtaining the same object, a tentative method, similar to that now usually employed in extracting the square root, but taking into account the second term of the quadratic, was adopted, and was afterwards extended to equations of higher orders, a method which was not only tentative but tedious and barren.

2. The actual solution of the quadratic equation in general algebraical symbols was of the greatest importance; the interpretation of the two roots which satisfied the equation, the consideration of these roots when imaginary, and the obvious relations subsisting between the roots and the coefficients of the given equation, much more than the solution of the cubic, and of the biquadratic which soon followed, tended to replace a string of unconnected devices by a true analysis possessing connexion and symmetry. Thus, in the hands of Harriot and Descartes, algebra ceased to be a system of artifices and became a science.

This science thus opened was cultivated in this country by Wallis, Newton, Cotes, Waring, and Thomas Simpson, and abroad by many, amongst whom we distinguish Bezout, Tchirnhausen, Euler, and Lagrange. Difficulties, however, remained, many of which have been

successfully combated by men of the present day; and the next generation will probably find the subject far from exhausted.

There probably exists no branch of pure analysis on which the exercise of close reflection is more calculated to improve the student in precision and elegance of research than the Theory of Equations.

3. The first step towards classifying equations consists in arranging the terms, according to integer powers of the unknown quantity, commencing with that having the highest index, and descending uniformly to the absolute term, or that which does not involve the unknown quantity; all such terms being placed on the left-hand side of the sign of equality, zero will alone remain on the right.

If x denote the unknown quantity, the left-hand member thus prepared is called a rational and integer function of x, because this symbol is then not affected by either fractional or negative indices.

Further, it adds to the simplicity, while it does not diminish the generality of an equation, to divide the whole by the coefficient of the first term, or that which involves the highest power of x.

Example 1. Arrange the terms of the equation

$$2x + a = \frac{ax - 2b}{x + 2a}.$$

Result: $x^2 + 2ax + (a^2 + b) = 0.$

The given equation, we thus learn, is of the second order, *that* being the highest power to which x is raised in the arranged equation.

Example 2. Arrange the terms of the equation

$$\left\{x + \alpha + (x^2 + 2\alpha x + \beta)^{\frac{1}{2}}\right\}^3 + \left\{x + \alpha - (x^2 + 2\alpha x + \beta)^{\frac{1}{2}}\right\}^3 + \gamma = 0.$$

By taking the actual cubes, the irrational parts disappear in the addition of the two; we thus obtain, first

$$2(x + \alpha)^3 + 6(x + \alpha)(x^2 + 2\alpha x + \beta) + \gamma = 0,$$

and from thence, by further reduction, the result, which is

$$x^3 + 3\alpha x^2 + \frac{1}{4}(9\alpha^2 + 3\beta)x + \frac{1}{4}\left(\alpha^3 + 3\alpha\beta + \frac{\gamma}{2}\right) = 0,$$

therefore the equation proposed is of the 3rd order.

Example 3. Arrange the terms of the equation

$$(x^2 + \alpha x + \beta)^2 = \gamma.(x + \delta)^2.$$

Result:

$$x^4 + 2\alpha x^3 + (\alpha^2 + 2\beta - \gamma)x^2 + 2(\alpha\beta - \gamma\delta)x + (\beta^2 - \gamma\delta^2) = 0,$$

an equation of the fourth degree.

4. When the equation is thus arranged, the index of the first term (which is the highest) marks, as it has been variously termed, the order, degree, or dimension of the equation, and also, of the rational function which constitutes its left member.

Proposition I.

Positive values, so great, may be assigned to x, that the corresponding values of x^n shall be incomparably greater than those of any given rational integer function of x of dimensions lower than n.

Let $Ax^m + Bx^{m-1} + Cx^{m-2} + \ldots\ldots + Px + Q$ be the given function, where m is less than n.

This function for abridgment we shall denote by $\phi(x)$, and suppose M to be *numerically* the greatest of the coefficients A, B, C . . . . P, Q, if they are unequal, or one of them if equal.

Then it is plain that
$$\begin{aligned} Mx^m &> \text{ or } = Ax^m, \\ Mx^{m-1} &> \text{ or } = Bx^{m-1}, \\ Mx^{m-2} &> \text{ or } = Cx^{m-2}. \\ &\text{\&c. \&c.} \end{aligned}$$

In the right-hand members of these relations the signs of the quantities which constitute them may be at variance, some perhaps positive, others negative; if so, this would only strengthen our conclusion formed by taking the sums, viz.,

$$Mx^m + Mx^{m-1} + Mx^{m-2} + \text{\&c.} + Mx + M >,$$
$$\text{or, } = Ax^m + Bx^{m-1} + Cx^{m-2} + \ldots + Px + Q;$$

hence, $$M.\frac{x^{m+1}-1}{x-1} > \text{ or } = \phi(x);$$

and supposing $x > 1$, it obviously follows that

$$M.\frac{x^{m+1}}{x-1} > \phi(x).$$

Now, $$x^n : \frac{Mx^{m+1}}{x-1} :: x^{n-m-1} : \frac{M}{x-1}.$$

If $x - 1$, and therefore x be exceedingly great, then $\frac{M}{x-1}$ is very small, converging to zero, while x^{n-m-1} is either unity (when $m = n - 1$), or exceedingly great (when $m < n - 1$); therefore x^{n-m-1} is incomparably greater than $\frac{M}{x-1}$, and by proportion x^n must be similarly far greater than $\frac{Mx^{m+1}}{x-1}$, and for a stronger reason greater than $\phi(x)$.

To make $x^{n-m-1} > \frac{M}{x-1}$, we have only to suppose $(x-1)^{n-m-1} = \frac{M}{x-1}$, which gives $x = 1 + M^{\frac{1}{n-m}}$, this value and much more any greater value of x will render $x^n > \phi(x)$.

Example. Assign to x such a value that $x^3 > 7x^2 + 6x + 5$. Here $M = 7$, $n - m = 1$, hence $x = 8$, or upwards, will have the desired effect.

5. When the terms of an equation are properly arranged (Art. 3.) we may assign to x values *so great*, that any term may be made far greater than the sum of all the succeeding; this we have seen in Art. 4.; and with respect to the terms *preceding*, we have an analogous theorem, viz.

Proposition II.

Positive values so small may be assigned to x, that the corresponding values of x^n shall be incomparably greater than those of any function of x consisting of powers of x higher than the nth.

Let $Ax^m + Bx^{m+1} + Cx^{m+2} + \ldots\ldots Px^{q-1} + Qx^q$ be the given function, where $m > n$.

Denoting this function by $\phi(x)$, and supposing M to be the greatest coefficient, we have as before,

$$Mx^m + Mx^{m+1} + Mx^{m+2} \ldots + Mx^{q-1} + Mx^q > \text{ or } = \phi(x).$$

Hence, if we continue the left hand member to infinity, which will be convergent when $x < 1$, we shall have

$$\frac{Mx^m}{1-x} > \phi(x).$$

Now, $$x^n : \frac{Mx^m}{1-x} :: 1 - x : Mx^{m-n},$$

as x diminishes between 1 & 0, x^{m-n} also diminishes, converging to zero, while $1 - x$ increases, converging to unity; therefore the ratio $1 - x : Mx^{m-n}$ continually increases, and when x is sufficiently small, may be made $>$ than any assigned ratio. Hence, x^n is then incomparably greater than $\frac{Mx^m}{1-x}$, and *à fortiori* than $\phi(x)$.

Put $x = \frac{1}{y}$, then $1 - x : Mx^{m-n} :: y - 1 : \frac{M}{y^{m-n-1}}$; and now to make $y - 1 > \frac{M}{y^{m-n-1}}$, put it $= \frac{M}{(y-1)^{m-n-1}}$, which gives $y = 1 + M^{\frac{1}{m-n}}$; therefore $\frac{1}{1 + M^{\frac{1}{m-n}}}$, and *à fortiori*, any smaller fraction when put for x will render $x^n > \phi(x)$.

Example. Assign to x such a value that $3x > 8x^2 + 9x^3$.

Divide, first by 3, then $M = 3$, $m - n = 1$, hence $x = \frac{1}{4}$, or any smaller fraction will answer the condition required.

6. These two propositions teach us what terms in a rational function are the most important when values exceedingly great or small are assigned to x, viz.: in the former case, that which contains the highest power of x; in the latter, that which contains the lowest.

To obtain a clearer insight into the reasoning of the following articles, it is advisable that the student should practise the *tabulating* of formulæ, or, which is the same thing, the *tracing* of functions.

If, for instance, $\phi(x)$ is a function or formula to be tabulated, put for x successively...$-5h, -4h, -3h, -2h, -h, 0, h, 2h, 3h, 4h, 5h\ldots.$ and register the corresponding values of $\phi(x)$: the greater the number of terms to the left and right of zero in this arithmetical progression, and the smaller the common difference h, the more perfect will be the table.

Example 1. Tabulate the formula $x^2-5\,x=\phi\,(x)$

I.	$x=$	5	4	3	2	1	0	-1	-2	-3	-4	-5 &c.
	$\phi\,(x)=$	0	-4	-6	-6	-4	0	6	14	24	36	50
II.	$x=$	$\frac{5}{2}$	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2	$-\frac{5}{2}$
	$\phi\,(x)=$	$-\frac{25}{4}$	-6	$-\frac{21}{4}$	-4	$-\frac{9}{4}$	0	$\frac{11}{4}$	6	$\frac{39}{4}$	14	$\frac{75}{4}$

Example (2.) Tabulate $x^3-3\,x^2+2\,x=\phi\,(x)$

Example (3.) .... $x^4-4\,x=\phi\,(x)$

Example (4.) .... $\dfrac{x\,(x-2)}{x-1}=\phi\,(x)$

In the table marked I. the value of h is 1, and in that marked II. it is $\frac{1}{2}$; and an inspection of the consecutive values of $\phi\,(x)$, in both tables, from $x=2$ to $x=-2$, will show how much nearer these values are, in the table where h has the smaller value.

7. In thus tabulating $\phi\,(x)$, if α were any particular value assigned to x, the next value in the series would be $\alpha+h$, and the result in this case, namely, $\phi\,(\alpha+h)$, may be derived from the preceding result $\phi\,(\alpha)$ by the following theorem.

Proposition III.

Let $\phi'(\alpha)$ be that rational and integer function of α, which is derived from $\phi\,(\alpha)$ by multiplying each term of the latter by its index, and then diminishing the index by unity, each term retaining its proper sign.

Let $\phi''(\alpha)$ be that which is derived in the same manner from $\phi'(\alpha)$

$\phi'''(\alpha)$ .... $\phi''(\alpha)$

&c.

Then shall

$$\phi\,(\alpha+h)=\phi\,(\alpha)+\phi'(\alpha).h+\phi''\,(\alpha).\frac{h^2}{1.2}$$

$$+\phi'''(\alpha).\frac{h^3}{1.2.3}+\ldots\ldots.\phi^{\prime(n)}\,(\alpha).\frac{h^n}{1.2.3\ldots n}$$

For let the given function which is represented by $\phi\,(x)$ be

$$x^n+\mathrm{A}\,x^{n-1}+\mathrm{B}\,x^{n-2}+\ldots.+\mathrm{P}\,x+\mathrm{Q}=\phi\,(x)$$

Then

$$\phi\,(\alpha+h)=(\alpha+h)^n+\mathrm{A}\,(\alpha+h)^{n-1}+\mathrm{B}\,(\alpha+h)^{n-2}+\ldots+\mathrm{P}\,(\alpha+h)+\mathrm{Q}.$$

Now expand each term by the binomial theorem, and let the whole be so arranged that like power of h stand in the same vertical columns.

Hence

$$\phi\,(\alpha+h)=\alpha^n+n\,\alpha^{n-1}.h+n.(n-1)\,\alpha^{n-2}.\ldots\frac{h^2}{1.2}+\ldots.n\,\alpha\,h^{n-1}+h^n$$

$$+\mathrm{A}\alpha^{n-1}+(n-1)\mathrm{A}\alpha^{n-2}.h+(n-1)(n-2)\mathrm{A}\alpha^{n-3}\frac{h^2}{1.2}+\ldots\mathrm{A}\,h^{n-1}$$

$$+\mathrm{B}\,\alpha^{n-2}+(n-2)\mathrm{B}\alpha^{n-3}.h+(n-2)(n-3)\mathrm{B}\alpha^{n-4}\frac{h^2}{1.2}+\ldots$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\mathrm{P}\,\alpha+\mathrm{P}.h$$

$$+\mathrm{Q}$$

The first vertical column does not contain h, its value is $a^n+A a^{n-1}+B a^{n-2}+\dots+P a+Q$, or which is the same thing $\phi(a)$

The coefficient of h in the second vertical column is

$$n a^{n-1}+(n-1)A a^{n-2}+(n-2)B a^{n-3}+\dots+P$$

which is derived from $\phi(a)$ by multiplying each term by its index, and diminishing that index by unity; it is therefore $\phi'(a)$, as defined in the announcement of this proposition.

And in the same way the coefficient of $\frac{h^2}{1.2}$, in the third vertical column, is derived from $\phi'(a)$ in the same manner that the latter was derived from $\phi(a)$; it is therefore $\phi''(a)$, by the notation agreed upon in the proposition.

We have thus a simple and condensed mode of forming $\phi(a+h)$ from $\phi(a)$, and arranging it according to the ascending powers of h.

$\phi'(a)$ is called the first derived function of $\phi(a)$

$\phi''(a)$ is the first derived of $\phi'(a)$, or the second derived of $\phi(a)$

$\phi'''(a)$ is the third derived of $\phi(a)$, and so on.

It is obvious from the proof that the general formula of this article holds true whenever $\phi(x)$ is an aggregation of terms involving any powers of x, whether negative or fractional, but in the theory of equations those powers are most commonly positive integers.

Example. Let $\phi(x)=x^3-3x^2+2x$

When $x=a=1$ $\phi(x)=\phi(a)=0$; it is required to find the value of $\phi(x)$ when $x=\frac{11}{10}=a+h$, where $h=.\frac{1}{10}$

$$\begin{aligned}
&\text{We have}\dots\dots\dots \phi\ (a)=a^3-3a^2+2a=0\\
&\text{1st derived function } \phi'\ (a)=3a^2-6a+2=-1\\
&\text{2nd derived}\dots\dots\dots \phi''\ (a)=6a-6 \qquad =0\\
&\text{3rd derived}\dots\dots\dots \phi'''(a)=6
\end{aligned}$$

$$\text{Hence } \phi(a+h)=\phi(a)+\phi'(a).h+\phi''(a).\frac{h^2}{1.2}+\phi'''(a).\frac{h^3}{1.2.3}$$

$$=0-\frac{1}{10}+0+\frac{1}{10^3}$$

$$=-.099$$

8. In the last example it was seen that when $x=1$ $\phi(x)=0$, and when $x=1.1$ $\phi(x)=-.099$ a value differing but little from zero, which was the former value. If h were taken, still smaller for instance, if $h=\frac{1}{100}$, that is, if $x=1.01$, the resulting value of $\phi(x)$ would be still nearer zero than before, and by taking h sufficiently small, we make the result to be as near that produced by substituting unity for x as we may desire; this will be generally proved by the following

Proposition IV.

Every rational and integer function of x will produce an uninterrupted series of values, which shall be nearer to each other than by any

assigned difference, however small, by merely assigning to x a series of values with small consecutive differences.

For let α be one of the values assigned to x in the given rational function $\phi(x)$, and $\alpha+h$ the next consecutive value of x; the corresponding results by the substitution of these values for x are $\phi(\alpha)$ and $\phi(\alpha+h)$ respectively. Now by Prop. III.

$$\phi(\alpha+h)=\phi(\alpha)+\phi'(\alpha).h+\phi''(\alpha).\frac{h^2}{1.2}+\phi'''(\alpha)\frac{h^3}{1.2.3}+\ldots.$$

$$+\phi'''^{(n)}(\alpha).\frac{h^n}{1.2.3..n}$$

$$=\phi(\alpha)+\phi'(\alpha)\left\{h+\frac{\phi''(\alpha)}{\phi'\alpha}.\frac{h^2}{1.2}+\frac{\phi'''(\alpha)}{\phi'(\alpha)}.\frac{h^3}{1.2.3}+\&c.\right\}$$

the meaning of the accented functions being the same as that described in the preceding article.

The part of this last expression enclosed between the brackets is a rational and integer function of h, and all the coefficients of the powers of h are finite numerical quantities, except when $\phi'(\alpha)=0$, which, at present, we shall suppose not to be the case.

Now by Prop. II. a value so small may be assigned to h that the first term, viz. h, may be made incomparably greater than the amount of all the succeeding, viz. $\frac{\phi''(\alpha)}{\phi'(\alpha)}.\frac{h^2}{1.2}+\frac{\phi'''(\alpha)}{\phi'(\alpha)}.\frac{h^3}{1.2.3}+\&c.$

Hence the difference between $\phi(\alpha+h)$ and $\phi(\alpha)$ may be made to differ by as small a shade as we please from $\phi'(\alpha).h$; and it is clear that, by giving very small values to h, this may be made less than any assigned quantity.

But if $\phi'(\alpha)$ vanished, a similar reasoning would lead us to the same certain conclusion, for then

$$\phi(\alpha+h)=\phi(\alpha)+\frac{\phi''(\alpha)}{2}\left\{h^2+\frac{\phi'''(\alpha)}{\phi(\alpha)}.\frac{h^3}{3}+\&c.\right\}$$

and the terms after h^2 between the brackets may, by Prop. II., be made to bear as small a ratio to the first h^2 as we please, so that the whole may be represented by kh^2, where k is exceedingly near to unity, hence $\phi(\alpha+h)-\phi(\alpha)=\frac{k}{2}.\phi''(\alpha).h^2$, which of course can be made less than any assigned quantity.

And in the same manner we could continue to reason if $\phi''(\alpha)$ vanished as well as $\phi'(\alpha)$, and so on.

But may not all the accented quantities vanish? No, for the last term $\phi'''^{(n)}(\alpha)\frac{h^n}{1.2..n}$ is exactly h^n, as will be immediately seen by referring to the manner in which the functions $\phi'(\alpha)$, $\phi''(\alpha)$, &c., are formed.

Therefore, by taking h sufficiently small, $\phi(\alpha+h)$ *must differ* from $\phi(\alpha)$, and that difference may be made less than any assigned quantity.

This proposition shows that $\phi(x)$, which is here used for a rational integer function of x, is perfectly continuous, that is, while the results are always real, the difference $\phi(\alpha+h)-\phi(\alpha)$ and $\phi(\alpha-h)-\phi(\alpha)$

converge to zero, as h is made to diminish continually towards the same, whatever α may be.

9. In assigning to x an indefinite series of values which consecutively differ from each other by only very small quantities, the resulting values of $\phi(\alpha)$, when tabulated, present to the view the nature, if we may so speak, of the function; these results may sometimes go on successively increasing, at another stage they may commence to diminish, and continue thence to diminish, up to a third stage, where they again may recommence to increase, and so on, or they may go on continually increasing or diminishing. At the stage at which the function ceases to increase and commences to diminish, it is said to have acquired there a maximum value, which, however, is not to be taken as absolutely the greatest of all the values in the table, but only of those which immediately precede and succeed it in the determined order of tabulation. In the same sense the function is said to have acquired a minimum value when, after decreasing, it has arrived at a stage whence it commences to increase. The least of all the minimum values of which the function is susceptible is called the absolute minimum, and the greatest of the maximum values, the absolute maximum; the true characters by which we can recognise when a function has really attained its maxima or minima values, will be given a little further on; the preceding observations will however materially tend to the comprehension of the full force of the following

PROPOSITION V.

If A and C are two different resulting values of a rational and integer function $\phi(x)$, which are produced when two quantities α, γ are substituted for x, then if B be any number chosen intermediate between A and C, it will be the result of the substitution of some quantity β, for x, which is itself intermediate between α and γ.

Let α be the *least* of the two quantities α, γ, including under this designation a negative number relatively to zero, or a positive; in other words, let $\gamma-\alpha$ be positive.

Again, suppose first that the result A is less than the result C.

Now when $\alpha+h$, $\alpha+2h$, $\alpha+3h$, &c., are substituted in the function for x, h being a very small common difference, it is *possible* that the results, instead of increasing and thereby approaching to C, may commence with diminishing; but they cannot continue so to do in the interval from $x=\alpha$ to $x=\gamma$, for then they could not ultimately produce the result C without a breach of continuity; which is contrary to Prop. IV.

They must therefore either increase all along between the abovementioned limits, or, if they commence with diminishing, they must arrive at one or several minima values: from the last of which (that is, the minimum for which the corresponding value of x is nearest to γ) the results necessarily increase uninterruptedly up to C when $x=\gamma$. All values therefore intermediate between the absolute minimum and C, are then passed through at least *once*, and therefore by a stronger reason all values intermediate between A and C, are necessarily passed through once or oftener in the actual limits of the series of values given to x; some quantity β between these limits (or it may be several) necessarily corresponds to the result B.

A similar train of reasoning would manifestly apply to that case where C is the least of the two given results.

10. By the theorems announced in the preceding propositions, we are enabled to trace rational functions, and observe the values of which they are susceptible; we can thus know whether a possible or impossible condition is imposed when such a value is to be assigned to x that $\phi(x)$ may acquire a certain given value; for uniformity this given value is made to be generally zero; if it were otherwise, as γ, we may bring it to zero by transposing γ, that is, subtracting γ from $\phi(x)$ (vide Art. 3); should the condition thus be possible, it must be verifiable by some value or values of x from $-\infty$ to $+\infty$ (which are usually termed real); and should it be impossible, it is a curious and interesting fact that an algebraical value of x, of the form $\alpha+\beta\sqrt{-1}$ (called imaginary), where α and β are *real* quantities, may then be found, which shall fulfil the condition which it was impossible to fulfil by real quantities; this part of the subject we now proceed to consider, first observing that the quantity, real or imaginary, which when put for x would verify, or render identical, the equation $\phi(x)=0$ is called *a* root of that equation. We use the indefinite article, for there is nothing in the preceding investigations to show that there may not be several different roots, and, farther on, we shall see that such there generally are.

Proposition VI.

Every equation of odd dimensions has a real root.

Let $\phi(x)=x^n+ax^{n-1}+bx^{n-2}+\ldots+px+q=0$ be the given equation where n is an odd number.

Now we may assign to x a value so great and positive that $x^n > ax^{n-1}+bx^{n-2}+\ldots$ &c. in quantity, by Proposition I.

Hence the function $x^n+ax^{n-1}+bx^{n-2}+$ &c. is then positive. Let P be this value of x, and $+A$ the resulting value of $\phi(x)$.

Again put $x=-y$, and observing that the odd powers of negative quantities are negative, and the even powers positive, the function then becomes

$$-(y^n-ay^{n-1}+by^{n-2}-\text{\&c.})$$

Now by the same proposition a value so great may be assigned to y that in quantity $y^n>ay^{n-1}-by^{n-2}+$ &c., and therefore

$$-(y^n-ay^{n-1}+by^{n-2}-\text{\&c.}) \text{ is negative.}$$

But this great positive value of y is equivalent to a great negative value of x, which therefore renders $x^n+ax^{n-1}+$ &c. negative. Let $-Q$ be this negative value of x, and $-B$ the resulting value of $\phi(x)$.

Hence by Prop. V., there exists a real quantity, intermediate to P and $-Q$, which shall render $\phi(x)$ any quantity intermediate to A and $-B$; let this intermediate stage be zero, and the corresponding intermediate quantity between P and $-Q$ by which it is produced is then a root of the given equation $\phi(x)=0$.

Cor. We may further easily perceive that this root has a sign contrary to q the last term of the function.

For if we put $x=0$ the result is q: suppose this positive, then put $x=-Q$, and the result is $-B$, which is negative, hence the intermediate result zero is producible by a quantity between O: and $-Q$, that

is by a negative root; in the same manner, if q be negative, there must be a real root between O and $+P$, since the latter gives a positive result.

Proposition VII.

Every equation of even dimensions, of which the last term is negative, has at least two real roots, one of which is positive, and the other negative.

Let $\phi(x)=x^n+ax^{n-1}+bx^{n-2}+\ldots+px-q=0$ be the equation.

Let x be taken so great and positive that $x^n>ax^{n-1}+bx^{n-2}$ &c.; abstracting from the sign of the latter, let P be this value of x, the resulting value of $\phi(x)$ must be some positive quantity $+A$.

Again put $x=-y$, then

$$\phi(x)=y^n-ay^{n-1}+by^{n-2}\ldots-py-q$$

Let y be taken so great and positive that $y^n>ay^{n-1}-by^{n-2}$ &c., and suppose Q to be this value of y, and the resulting value of $\phi(x)$ is then obviously also a positive quantity as $+B$.

Lastly, let zero be put for x, the result is the negative quantity $-q$.

Therefore, first, between 0 and $+P$ there must exist a real root by Prop. V.; and, secondly, another real root must lie between O and $-Q$; the former is evidently positive, the latter negative.

11. Equations in which the last term is very small, compared with the coefficients of the other terms, have always real roots, which are also very small, as will be seen by the following

Proposition VIII.

Let an equation, $\phi(x)=0$ be divided by the coefficient of x, so as to be reduced to the form

$$-\kappa+x+ax^2+bx^3+cx^4+\&c.=0=\phi(x)$$

Suppose that κ is less than $\frac{1}{4(M+1)}$ where M is the greatest coefficient, then the equation has a real root less than $\frac{1}{2(M+1)}$, and of a sign contrary to the absolute term $-\kappa$.

First, suppose $-\kappa$ to be actually negative, then $x=0$ renders $\phi(x)=-\kappa$ a negative quantity.

Again, a value α for x would manifestly render $\phi(x)$ positive, if it could make $-\kappa+x-Mx^2-Mx^3-Mx^4$, &c., ad inf. to be zero, supposing this value of α to be less than unity.

Put therefore $\qquad -\kappa+\alpha-\frac{M\alpha^2}{1-\alpha}=0$

Hence $\qquad (M+1)\alpha^2-(1+\kappa)\alpha+\kappa=0$

$$\alpha=\frac{1+\kappa}{2(1+M)}-\sqrt{\left\{\left(\frac{1+\kappa}{2(1+M)}\right)^2-\frac{\kappa}{1+M}\right\}}$$

$$=\frac{1}{2(1+M)}\{1+\kappa-\sqrt{\{1+2\kappa+\kappa^2-4\kappa(1+M)\}}$$

This value of α is obviously real, since $4\kappa(1+M)<1$, and the part

under the radical sign is greater than $\sqrt{\kappa^2+2\kappa}$, and therefore $>\kappa$, therefore α is less than $\frac{1}{2(1+M)}$, and since α put for x makes $\phi(x)$ positive, and zero makes it negative, there is a real root between 0 and α, and therefore less than $\frac{1}{2(1+M)}$.

In like manner if the absolute term were κ instead of $-\kappa$, putting $x=-y$, and changing all the signs in the resulting equation, we should find y less than $\frac{1}{2(1+M)}$, and therefore x would be between zero and $-\frac{1}{2(1+M)}$.

PROPOSITION IX.

12. If such a value may be assigned to x as shall make a rational function $\phi(x)$ to become a maximum or minimum, this value will be a root of the derived equation $\phi'(x)=0$.

Let α be such a value, and suppose the function to be then a minimum, we must have

$$\phi(\alpha)<\phi(\alpha+h) \text{ and } <\phi(\alpha-h);$$

therefore $\phi(\alpha+h)-\phi(\alpha)$ and $\phi(\alpha-h)-\phi(\alpha)$ must have the same sign when h is sufficiently small.

Or by Proposition III.

$$\left.\begin{array}{l}\phi'(\alpha).h+\phi''(x).\frac{h^2}{1.2}+\phi''(\alpha).\frac{h^3}{1.2.3}+\&c. \\ \text{and}-\phi'(\alpha).h+\phi''(\alpha).\frac{h^2}{1.2}-\phi'''(\alpha).\frac{h^3}{1.2.3}+\&c.\end{array}\right\}\begin{array}{c}\text{must have like}\\ \text{signs.}\end{array}$$

Now by Proposition II., a value so small may be assigned to h that the first term in each series may become much greater than the sum of all the others, consequently these series cannot have like signs unless $\phi'(\alpha)=0$, that is α must be a root of the derived equation $\phi'(x)=0$.

The sign of each series will then be the same as that of $\phi''(\alpha)$, this must therefore be a positive quantity for the minimum; and the same proof shows that it would be negative for a maximum, unless it were altogether to vanish, then the same reasoning would show that α was also a root of the 3rd derived equation, and the same criterion might be applied to the 4th derived function $\phi''(\alpha)$ to determine whether $\phi(\alpha)$ was a maximum or minimum, and so on.

Example. Let $\phi(x)=x^2+5x$

then $\phi'(x)=2x+5$

$\phi''(x)=2$

The sign of $\phi''(x)$ is positive, and therefore $\phi(x)$ is susceptible of only a minimum, and not of a maximum value; this minimum occurs when $\phi'(x)=0$ or $x=-\frac{5}{2}$, and its amount is $-\frac{25}{4}$.

Example II. Let $\phi(x)=x^3-27x$

$\phi'(x)=3x^2-27 \quad \phi''(x)=6x.$

If $\phi'(x)=0$ we obtain $x=+3$, or -3, the former making $\phi(x)$ a minimum, the latter a maximum.

Put, for instance, -2, -3, -4 successively for x, and register the corresponding values of $\phi(x)$; they are $+46$, $+54$, $+44$.

Thus we perceive $x=-3$ makes $\phi(x)$ a maximum relative to the values preceding and succeeding it, but not an *absolute* maximum, since x may be made so great and positive, that x^3 may be incomparably $>27x$; for a similar reason $x=3$ would not produce an absolute minimum.

In short, no rational function of odd dimensions is susceptible of an absolute maximum or minimum, as appears by the demonstration of Proposition VI.

13. But every function of even dimensions is susceptible of an absolute minimum, as appears by the following

Proposition X.

Let $x^n+ax^{n-1}+bx^{n-2}+$ &c. $=\phi(x)$ be a rational function of even dimensions; and M the greatest coefficient abstracting from its sign.

No real value can be assigned to x which shall render $\phi(x)$ less than the least of the two quantities $-\mathrm{M}n$ and $-\mathrm{M}\{(n-1)\mathrm{M}\}^{n-1}$, that negative quantity which is most remote from zero being considered least.

For n being even, whether to x we assign a positive or negative value, we have $\phi(x)>x^n-\mathrm{M}\ \{x^{n-1}+x^{n-2}+\ldots.x^2+x+1\}$. Now if we assign to x first a value less than unity, then evidently

$$1+x+x^2+\ldots.x^{n-1}<n$$

therefore $$\phi(x)>x^n-\mathrm{M}n$$

and x^n being positive, it follows that $\phi(x)>-\mathrm{M}n$, when for x we put any quantity between $+1$ and -1, inclusive.

Again, if $x>1$, then $nx^{n-1}>x^{n-1}+x^{n-2}+x^{n-3}+x^2+x+1$;

therefore $$\phi(x)>x^n-n\mathrm{M}x^{n-1}.$$

Now the minimum value of $x^n-n\mathrm{M}x^{n-1}$ is found by putting its derived function $nx^{n-2}\{x-(n-1)\mathrm{M}\}=0$, which can only be satisfied by $x=0$ or $x=(n-1)\mathrm{M}$, the latter giving the minimum, viz., $x^{n-1}(x-n\mathrm{M})=-\mathrm{M}x^{n-1}=-\mathrm{M}\{(n-1)\mathrm{M}\}^{n-1}$; that this is the minimum we are assured by consulting the sign of the second derived function.

Hence $\phi(x)>-\mathrm{M}\{(n-1)\mathrm{M}\}^{n-1}$ for all values of x from 1 to ∞ and from -1 to $-\infty$.

Thus it is easy to observe the nature of the continuity of the values acquired by functions of even dimensions. When x is very great and positive, $\phi(x)$ is great and positive; as x diminishes, $\phi(x)$ also diminishes towards a minimum, to which it afterwards arrives. If this be not an absolute minimum, and we continue to diminish, x, $\phi(x)$ then increases towards a relative maximum, after which it must again diminish, and so on, until it arrives at an absolute minimum, after which, if we continue to diminish x, so as to make it approach $-\infty$, $\phi(x)$ increases towards $+\infty$ beyond any assignable limit.

Let us suppose γ to be the absolute minimum of $\phi(x)$, and γ' be any quantity less than γ, the equation $\phi(x)=\gamma'$ is not verifiable by

any real value of x from $-\infty$ to $+\infty$, in other words, the equation $\phi(x)-\gamma'=0$ has no real root.

Example. Let $\phi(x)=x^2+ax$

Then $\phi'(x)=2x+a$ $\phi''(x)=2$, we have therefore an absolute (because the only) minimum, when $x=-\frac{a}{2}$; therefore $\gamma=-\frac{a^2}{4}$: hence if $x^2+ax=\gamma'$, where γ' is less than $-\frac{a^2}{4}$, the equation will have no real root; the term *less*, when the sign is retained, indicates that quantity which, subtracted from the other, would give a positive remainder.

Proposition XI.

Let $\phi(x)$ be a rational function of x, and $\phi'(x)$ its derived function, and γ a quantity introduced.

Divide $\phi(x)-\gamma$ by $\phi'(x)$ in the manner of finding the greatest common measure; the remainder, which will be of lower dimensions than $\phi'(x)$, being the next divisor, and $\phi'(x)$ divided.

Continue this process until x disappears in the final remainder, which is a function of γ, as $F(\gamma)$.

Then the real roots of the equation $F(\gamma)=0$ will be the general maxima and minima values of $\phi(x)$, and when there is an absolute minimum it is the least root of this equation, the term less having reference to the sign.

For the value of γ rendering the remainder equal to zero, it follows that $\phi(x)-\gamma$, and $\phi'(x)$ have a common measure, such as $x-\alpha$, which, if the corresponding quotients were P and Q, would give

$$\begin{aligned}\phi(x)-\gamma&=(x-\alpha)P\\ \phi'(x)\quad&=(x-\alpha)Q,\end{aligned}$$

and putting α for x in the identities,

we have
$$\begin{aligned}\phi(\alpha)-\gamma&=0\\ \phi'(\alpha)\quad&=0\end{aligned}$$

the latter equation showing that $\phi(\alpha)$ is a minimum or maximum, and the former that γ is the value of this minimum or maximum. Consequently the absolute minimum, if such existed, would be the least value of γ.

The process described in this Proposition is to be followed exactly similarly to that of finding the greatest common measure of algebraical quantities, namely, by introducing factors into the dividends to avoid fractional forms.

Example I. Let
$$\begin{aligned}\phi(x)&=x^2+ax\\ \phi'(x)&=2x+a\\ 2(\phi(x)-\gamma)&=2x^2+2ax-2\gamma\end{aligned}$$

Actual division
$$\begin{array}{r}2x+a\,)\,2x^2+2ax-2\gamma\,(x\\ \underline{2x^2+\ ax\qquad}\\ ax-2\gamma\end{array}$$

Multiplied by 2.
$$\begin{array}{r}2ax-4\gamma\,(a\\ \underline{2ax+a^2}\\ -a^2-4\gamma\end{array}$$

The final remainder is therefore (with sign changed), $4\gamma + a^2$, and if we put this $= 0$, we find the absolute minimum $\gamma = -\frac{a^2}{4}$.

Example 2. $\phi(x) = x^3 + ax$

$\phi'(x) = 3x^2 + a$

$$
\begin{array}{l}
3x^2 + a)3x^3 + 3ax3\gamma(x \\
\quad 3x^3 + ax \\
\hline
2ax - 3\gamma)6ax^2 + 2a^2(3x \\
\quad 6ax^2 - 9\gamma x \\
\hline
\quad 9\gamma x + 2a^2 \\
\quad 18a\gamma x + 4a^3(9\gamma \\
\quad 18a\gamma x - 27\gamma^2 \\
\hline
\quad 27\gamma^2 + 4a^3
\end{array}
$$

Thus the relative max. and min. are the roots of the equation

$$\left(\frac{\gamma}{2}\right)^2 + \left(\frac{a}{3}\right)^3 = 0.$$

In like manner we can find how many are maxima and how many minima, for $\phi''x$ being positive for a maximum, and negative for a minimum, we have only to proceed in the same manner for finding the greatest common measure of $\phi'(x)$ and $\phi''(x) - \beta$, until we arrive at a remainder independent of x, which therefore is a function of β, which, if we equate with zero, the number of positive values of β is the same as the number of minima, and that of the negative is the number of maxima values of the given function $\phi(x)$.

Example. $\phi(x) = x^3 - 3a^2x + b$

$\phi'(x) = 3(x^2 - a^2)$; $\phi''(x) = 6x$

Hence, rejecting the multipliers 3 and 6, we proceed as in finding the greatest common measure of $x^2 - a^2$ and $x - \beta$;

$$
\begin{array}{l}
(x - \beta)\,x^2 - a^2\,(x + \beta \\
\quad x^2 - \beta x \\
\hline
\quad \beta x - a^2 \\
\quad \beta x - \beta^2 \\
\hline
\quad \beta^2 - a^2.
\end{array}
$$

Now the equation $\beta^2 - a^2 = 0$ has two real roots, one positive, the other negative; hence, $\phi(x)$ admits one minimum and one maximum, and no more.

Proposition XII.

16. If by assigning a series of real values to x, any function of x as $\phi(x)$ attains a minimum value when $x = a$, it can be rendered yet less by making $x = \alpha + h + k\sqrt{-1}$ when h and k are very small; the sign of h being $+$ or $-$, according as the first derived function of $\phi(x)$, which does not vanish when $x = \alpha$ has the same sign with the succeeding derived function, or a contrary sign.

For if we separate the imaginary from the real part of $\phi(x + h + k\sqrt{-1})$ the expansion of this function is

$$\phi(x+h)-\phi''(x+h)\,.\,\frac{k^2}{1.2}+\phi^{iv}(x+h)\,.\,\frac{k^4}{1.2.3.4}-\&c.$$

$$+k\sqrt{-1}\left\{\phi'(x+h)-\phi'''(x+h)\,.\,\frac{k^2}{2.3}+\phi^{v}(x+h)\,.\,\frac{k^4}{2.3.4.5}-\&c.\right\}$$

Now suppose $x=\alpha$, then $\phi'(x)$, or $\phi'(\alpha)=0$, which is the condition necessary that $\phi(x)$ should be either a maximum or a minimum.

And it is peculiar to a minimum that $\phi''(x)$ or $\phi''(\alpha)$ may be positive unless it also vanishes, which we need not at present suppose to be the case (for the subsequent reasoning would then similarly apply to the next derived functions, since it would be then necessary that $\phi'''(\alpha)=0$ and $\phi^{iv}(\alpha)$ be positive; and if $\phi^{iv}(\alpha)$ also vanished, we should extend the same reasoning relative to the next two derived functions, and so on).

Now $\qquad \phi(x+h)=\phi(x)+\phi'(x)\,.\,h+\phi''(x)\,.\,\dfrac{h^2}{1.2}+\&c.$

and $\qquad \phi'(x+h)=\phi'(x)+\phi''(x)\,.\,h+\phi'''(x)\,.\,\dfrac{h^2}{1.2}+\&c.$

Making $x=\alpha$ in these expansions we find

$$\phi(\alpha+h)=\phi(\alpha)+\phi''(\alpha)\,.\,\frac{h^2}{1.2}+\&c.$$

$$\phi'(\alpha+h)=\phi''(\alpha)\,.\,h+\phi'''(\alpha)\,.\,\frac{h^2}{1.2}+\&c.$$

Therefore,

$$\phi(\alpha+h+k\sqrt{-1})=\phi(\alpha)+\phi''(\alpha)\,.\,\frac{h^2}{1.2}+\&c.$$

$$-\phi''(\alpha+h)\,.\,\frac{k^2}{1.2}+\phi^{iv}(\alpha+h)\,.\,\frac{k^4}{1.2.3.4}-\&c.$$

$$-k\sqrt{-1}\left\{\phi''(\alpha)\,.\,h+\phi'''(\alpha)\,.\,\frac{h^2}{1.2}+\&c.\right.$$

$$\left.-\phi'''(\alpha+h)\,.\,\frac{k^2}{2.3}+\phi^{v}(\alpha+h)\,.\,\frac{k^4}{2.3.4.5}-\&c.\right\}$$

Now in order that this function may be real, it is necessary that the quantity between the brackets may be zero.

Next suppose that h is taken very small, then

$$\phi''(\alpha)\,.\,h+\phi'''(\alpha)\,.\,\frac{h^2}{1.2}\;\&c.$$

may be made as small as we please, and putting it $=\kappa\,.\,\dfrac{\phi'''(\alpha+h)}{2.3}$ we have

$$-\kappa+k^2-\frac{\phi^{v}(\alpha+h)}{\phi'''(\alpha+h)}\,.\,\frac{k^4}{4.5}+\&c.=0.$$

Hence, by Prop. VIII., k^2 will have a real positive root to satisfy this equation, provided κ be essentially positive; that is, provided

$\phi''(\alpha).h + \phi'''(\alpha)\dfrac{h^2}{2.3}$ &c. has the same sign as $\phi'''(\alpha+h)$, or finally, that h has the same sign as $\dfrac{\phi''(\alpha)+\phi'''(\alpha).\dfrac{h^2}{2.3}\text{ \&c.},}{\phi'''(\alpha)+.\phi^{iv}(\alpha).h\text{ \&c.}}$; and since h is supposed very small, the sign of this fraction is the same as that of $\dfrac{\phi''(\alpha)}{\phi'''(\alpha)}$, therefore h must be positive or negative, according as $\phi''(\alpha)$ and $\phi'''(\alpha)$ have their signs like or unlike; and this we have supposed to be the case in the Proposition.

But k^2 having a real positive value, it follows that k has also a real positive and another negative value; therefore $\phi(\alpha + h + k\sqrt{-1})$ will be a real quantity.

Now $$\kappa = k^2 - \frac{\phi^{v}\ \alpha + k}{\phi'''(\alpha+h)}\cdot\frac{k^4}{4.5}\text{ \&c.} = \mathrm{A}k^2 \text{ suppose.}$$

And $$\frac{\kappa}{2.3} = \frac{\phi''(\alpha).h + \phi'''(\alpha).\dfrac{h^2}{2.3}}{\phi'''(\alpha+k)} = \frac{\mathrm{B}h}{6}\ldots.$$

whence we see that h is of the same order as k^2, the limit to which the fraction $\dfrac{k^2}{h}$ approaches as h diminishes, being $\dfrac{\mathrm{B}}{\mathrm{A}}$ or $\dfrac{6\phi''(\alpha)}{\phi'''(\alpha)}$, (putting $h = 0$ for the extreme case.)

Substitute for h its value $\dfrac{\mathrm{A}}{\mathrm{B}}.k^2$ in the expansion of $\phi(\alpha+h+k\sqrt{-1})$, and observing that the imaginary part has been made to vanish, it becomes

$$\phi(\alpha) - \phi''(\alpha+h).\frac{k^2}{1.2} + \phi^{iv}(\alpha+h).\frac{k^4}{1.2.3.4}. - \text{\&c.} + \frac{\mathrm{A}^2}{2\mathrm{B}^2}\phi''(\alpha).k^4 + \text{\&c.}$$

Now $$\phi''(\alpha+h) = \phi''(\alpha) + \phi'''(\alpha).\frac{\mathrm{A}}{\mathrm{B}}.k^2 + \text{\&c.}$$

Hence $$\phi(\alpha + h + k\sqrt{-1}) = \phi(\alpha) - \phi''(\alpha).\frac{k^2}{1.2} + \mathrm{D}.k^4 + \mathrm{E}.k^6 + \text{\&c.}$$

putting, for abridgment, $$\mathrm{D} = -\frac{\mathrm{A}}{2\mathrm{B}}.\phi''(\alpha) + \frac{\phi^{iv}(\alpha)}{1.2.3.4} + \frac{\mathrm{A}^2}{2\mathrm{B}^2}.\phi''(\alpha).$$
$$\mathrm{E} = \ldots\ldots\ldots\ldots\ldots$$

Now $\phi''(\alpha)$ being essentially positive, it follows that $\phi(\alpha+h+k\sqrt{-1}) < \phi(\alpha)$; and from what has been observed above, it is clear that we should find the same result if $\phi'''(\alpha) = 0$ and $\phi^{iv}(\alpha)$ were positive, which circumstance is then essential to a minimum.

Corollary. By a similar process it may be shown, that when a series of values are put for x in the function $\phi(x)$ until it reaches a maximum, it may be made to increase above this maximum by giving the corresponding value of x an imaginary increment similarly determined.

Example. Required to diminish the function

$$x^3-2x^2+x-2$$

below its minimum value.

Here
$$\phi(x)=x^3-2x^2+x-2$$
$$\phi'(x)=3x^2-4x+1$$
$$\phi''(x)=6x-4.$$

Put
$$\phi'(x)=0 \quad \text{or } x^2-\frac{4}{3}\cdot x=-\frac{1}{3}$$

whence
$$x=1 \quad \text{or } x=\frac{1}{3}$$

in the former case $\phi''(x)=2$, and in the latter $\phi''(x)=-2$; therefore the former only gives the minimum value of $\phi(x)$, which is $1-2+1-2=-2$.

To diminish the function below this we put $x=1+h+k\sqrt{-1}$.

Hence
$$\phi(x)=-2+(h+k\sqrt{-1})^2+(h+k\sqrt{-1})^3$$
$$=-2+h^2+h^3-k^2-3hk^2$$
$$+k\sqrt{-1}\,(2h+3h^2-k^2)$$

To make the imaginary part vanish we must have

$$k^2=2h+3h^2,$$

which will be small and positive if h be small and positive.

Then $\phi(x)$ becomes
$$-2+h^2+h^3-(2h+3h^2)(1+3h)$$
$$=-2-2h-8h^2-8h^3,$$

and since h is positive (the same sign as $\left.\frac{\phi''(x)}{\phi'''(x)}\right)$, it follows that this imaginary increment $h+k\sqrt{-1}$ given to that value (1) of x which corresponds to the minimum, succeeds in reducing the function below that minimum.

16. In the preceding proposition, as well as Proposition X., we have supposed that cases may occur in which not only the first derived function vanishes for a particular value of x, but also some of the succeeding ones for the same value: the following proposition will show under what circumstance any function of x and $m-1$ of its successive derived functions will all vanish for the same assigned value of x.

Proposition XIII.

When α is put for x in a function $f(x)$, and in its $m-1$ successive derived functions, should they all vanish it is necessary that $(x-\alpha)^m$ may be a factor of $f(x)$.

For if we put $\alpha+(x-\alpha)$ for x,

$\alpha^2+2\alpha(x-\alpha)+(x-\alpha)^2$ for x^2,

$\alpha^3+3\alpha^2(x-\alpha)+3\alpha^2(x-\alpha)+\alpha^3$ for x^3, &c.

it is clear that $f(x)$ will be reduced to the form

$$A_0+A_1(x-\alpha)+A_2(x-\alpha)^2+\ldots.A_{m-1}(x-\alpha)^{m-1}+A_m(x-\alpha)^m+\&c.=f(x)$$

Hence $$A_1+2A_2(x-\alpha)+3A_3(x+\alpha)^2+\ldots$$
$$\ldots(m-1)A_{m-1}(x-\alpha)^{m-2}+mA_m(x-\alpha)^{m-1}\ \&c.=f'(x).$$

Now when x is put equal to α, we suppose $f(x)$ and $f'(x)$ both to vanish, which manifestly requires that $A_0=0$ and $A_1=0$.

Therefore $$f(x)=A_2(x-\alpha)^2+A_3(x-\alpha)^3+\&c.$$
that is, $f(x)$ in this case has a factor of the form $(x\ \ \alpha)^2$.

If the next derived function, viz.
$$2A_2+6A_3(x-\alpha)+12A_4(x-\alpha)^2+\&c.$$
also vanish when $x=\alpha$, we must have $A_2=0$.

Therefore, $f(x)=A_3(x-\alpha)^3+A_4(x-\alpha)^4+\&c.$ must have $(x-\alpha)^3$ as a factor; and by continuing this process, it follows in general that when a function with its $m-1$ successive derived functions vanish for $x=\alpha$, then $(x-\alpha)^m$ must necessarily be a factor of $f(x)$.

Note. It is usual to say in this case that the equation $f(x)=0$ has m equal roots, by which it is meant, that $f(x)$ has m equal factors of the first degree, namely $(x-\alpha)\ (x-\alpha)\ (x-\alpha)\ldots\ldots m$ times.

Example. Let $f(x)=x^3-5x^2+8x-4$, which vanishes when $x=2$,
nence $f'(x)=3x^2-10x+8\ldots$ also vanishes for $x=2$,
$f''(x)=6x-10$ which does not then vanish.

In this, therefore, $f(x)$ has a factor $(x-2)^2$ or x^2-4x+4, which can be verified by actual division

$$\begin{array}{r|l} x^2-4x+4) & x^3-5x^2+8x-4\,(x-1 \\ & x^3-4x^2+4x \\ \hline & -\ x^2+4x-4 \\ & -\ x^2+4x-4 \\ \hline & 0 \end{array}$$

17. By Proposition XII. we have seen that where the series of values belonging to a function of x of any dimensions ceases to diminish when real quantities are substituted for x, a continuation of that series in the same direction, that is, diminishing towards negative infinity, is produced by the substitution of imaginary values; and the next theorem shows that this series thus prolonged may be continued on uninterruptedly, without ever *turning back*.

Proposition XIV.

An uninterruptedly decreasing series of values may be produced for any rational function of x of even dimensions, by the substitution of imaginary quantities $y+z\sqrt{-1}$ in the place of x.

Suppose $\phi(x)$ to be the proposed function, the following notation is adopted for abridgment:

$$\text{Let}\quad P=\phi(y)\ -\phi''(y).\frac{z^2}{1.2}+\phi^{\text{iv}}(y).\frac{z^4}{1.2.3.4}-\&c.$$
$$Q=\phi'(y).z-\phi'''(y).\frac{z^3}{1.2.3}+\phi^{\text{v}}(y).\frac{z^5}{1.2.3.4.5}-\&c$$
$$P'=\phi'(y)\ -\phi'''(y).\frac{z^2}{1.2}+\phi^{\text{v}}(y).\frac{z^4}{1.2.3.4}-\&c.$$
$$Q'=\phi''(y).z-\phi^{\text{iv}}(y).\frac{z^3}{1.2.3}+\phi^{\text{vi}}(y).\frac{z^5}{1.2.3.4.5}-\&c.$$

the accented quantities in the expansions denoting, as in Proposition III., the derived functions.

Put $y+z\sqrt{-1}$ for x, then $\phi(x)$ becomes

$$\phi(y+z\sqrt{-1}) = \text{P}+\text{Q}\sqrt{-1}\,;$$

and that this value may be real we must have $\text{Q}=0$, and as Q is of odd dimensions in y, we can find a real value of y for any value z; thus y becomes a function of z, and therefore also P is a function of z.

In giving particular values to z, and obtaining the corresponding values of y, which satisfy the equation $\text{Q}=0$, it is possible they may also make $\text{Q}'=0$, and then Q would be a minimum for this particular value of z; in that case, by the preceding proposition Q must be of the form $(y-\alpha)^2\,\psi(y)$, and $\psi(y)$ being of odd dimensions we can get a different value β for y, which, making $\psi(y)$ zero, will also make Q vanish without making Q′ disappear; the case of Q being a minimum at zero and therefore Q′ vanishing, we may thus dismiss altogether.

In the equation $\text{Q}=0$ suppose z to become $z+h$, and y to become $y+k$, Q becomes (as will be readily found by actual substitution)

$$\text{Q}+(\text{P}'h+\text{Q}'k)+\ \&\text{c., which must still equal zero;}$$

and taking h and k very small, we have

$$k = -\frac{\text{P}'}{\text{Q}'}\,.h+\text{A}h^2+\text{B}h^3\ \&\text{c.}$$

the first term of which is finite, since Q′ does not vanish.

In like manner we find that the real part P which expresses the value of $\phi(x)$ becomes

$$\text{P}+(\text{P}'k-\text{Q}'h)+\ \&\text{c.} = \text{P}-\frac{\text{P}'^2+\text{Q}'^2}{\text{Q}'}.h+\text{A}_1h^2+\text{B}_2h^3,\ \&\text{c}$$

and when h is small and of the same sign as Q′, the decrement of P is $\dfrac{\text{P}'^2+\text{Q}'^2}{\text{Q}'}.h$, the numerator being the sum of two squares, cannot vanish unless both $\text{P}'=0$ $\text{Q}'=0$; but the latter case we have seen how to avoid.

Thus the real functions of imaginary quantities do not admit of maxima or minima.

It is possible that Q′ may vanish without Q being a minimum; namely, if Q″ also $=0$ in this case, the diminution of P would go on without taking for y a different root from α, which we have done when Q was a minimum, for k would then be of the form $k=ph^3+qh^4$, &c., and the new value for $\phi(x)$ would be of the form $\text{P}-p_1h^3+q_1h^4$, &c.; but whenever Q for a certain value of z is a maximum or minimum, the diminution of P can only be effected by passing from α to a different real root, which, as we have seen, will always exist in such a case.

If we arrange Q and Q′ according to the powers of y, and proceed as in finding their greatest common measure, we ultimately must come to a remainder containing only z as $\text{F}(z)$, then all the values of z for which Q becomes a maximum or minimum are real roots of the equation $\text{F}(z)=0$: for the two equations $\text{Q}=0$ $\text{Q}'=0$ subsisting simultaneously, the equation $\text{F}(z)=0$ must also subsist, since it is merely the result of eliminating y between the former two.

Hence the following mode of forming an uninterruptedly decreasing series of values for $\phi(x)$ below the absolute minimum to which it arrives when real quantities are substituted for x.

Put for x $y+z\sqrt{-1}$, and taking z very small, we can by Proposition XII. diminish $\phi(x)$ below its absolute minimum, and as little below it as we please, so that the new chain of real values arising is a continuation below those resulting from the substitution of real quantities for x. Continue, according to the present Proposition, to diminish $\phi(x)$ by changing z through insensible degrees, the changes of y which correspond to small changes of z so as to preserve $Q=0$, will also be very small, until z arrives at a value which renders Q a minimum, and which is therefore a root of the equation $F(z)=0$; in this case, for a very small change of z, y has a finite or sensible change, and the real value of $\phi(x)$ or P has only a very small change, which we may take in the direction of its continued diminution, after which the corresponding changes of y and z will be again of the same order of magnitude, until z arrives at another value making Q a minimum, and therefore $F(z)=0$, when the process is continued as before.

It is moreover to be observed, that $\frac{Q}{z}=0$, from which the values of y are derived, would not contain an absolute term unless $\phi(x)$ contained some odd power of x; but this may always be effected by the system of substitutions given in Proposition XII.

Example. $\phi(x)=x^4-4x+6$,
when this function is at its minimum,

$$\phi'(x)=4(x^3-1)=0, \qquad \text{or } x=1;$$

thus $\phi(x)=+3$ is the absolute minimum by the substitution of real quantities for x.

Put $x=y+z\sqrt{-1}$, and $\phi(x)=P+Q\sqrt{-1}$
then $P=y^4-6y^2z^2+z^4-4y+6 \qquad Q=z\{4y^3-4yz^2-4\}=0.$

Hence, $Q'=4z\{3y^2-z^2\}$.

Now by taking z very small, we can get a real value for y, as in Proposition XII., by which P will be made less than the minimum value of $\phi(x)$.

Continue to increase z from zero upwards; Q', being supposed not to vanish, will have the same sign, and therefore P will continue to diminish: thus, let $z=1$, then the two values of y are very nearly equal, and to find when they become exactly equal, we proceed as for the greatest common measure:

$$\begin{array}{l}
3y^2-z^2)\,3y^3-3yz^2-3\,(y \\
\qquad\qquad \underline{3y^3-\ \ yz^2} \\
\qquad 2yz^2+3)\,6y^2z^2-2z^4\,(3y \\
\qquad\qquad\qquad \underline{6y^2z^2+9y} \\
\qquad\qquad\qquad\ \ 9y+2z^4 \\
\qquad\qquad\qquad\ \ 18yz^2+4z^6\,(9 \\
\qquad\qquad\qquad\ \ \underline{18yz^2+27} \\
\qquad\qquad\qquad\ \ 4z^6-27=F(z)=0
\end{array}$$

$$\text{or} \qquad z=\frac{\sqrt{3}}{\sqrt[3]{2}}=1.3 \text{ \&c.} \quad =\alpha.$$

then Q will be divisible by $(y-\alpha)^2$ and the quotient equated to zero will give a different value for y, with which the diminution of P is to be continued.

After passing this value of z, the case of Q' as minimum will not again occur, and therefore P will continue to diminish rapidly; thus,

let. $z=10$, then $Q=40\{y^3-100y-4\}$ or $y=10$ very nearly;

therefore, $P=-4(10)^4-40+6=-4036$ nearly;

and by continuing to increase z, the value of P continues to diminish towards negative infinity.

The reader may revert to this subject *of tracing* the real functions of imaginary quantities when in possession of methods for the numerical solution of equations.

18. Having thus discussed the mode of tracing functions by substituting for x quantities real and imaginary, we distinctly see that $\phi(x)$ is susceptible of any real value γ; and since for uniformity we take this quantity to be zero, it follows in other words, that when $\phi(x)$ is a rational and integer function of x, the equation $\phi(x)=0$ has always a root either real or of the form $\alpha+\beta\sqrt{-1}$ called imaginary; also if $\phi(\alpha+\beta\sqrt{-1})=P+Q\sqrt{-1}=0$, then it is easy to see that $\phi(\alpha-\beta\sqrt{-1})=P-Q\sqrt{-1}$, and we have seen that Q is always zero, therefore the first equation gives $P=0$, which renders $\phi(\alpha-\beta\sqrt{-1})=0$, and therefore imaginary roots are always couples of the form $\alpha\pm\beta\sqrt{-1}$.

Proposition XV.

An algebraical equation of n dimensions has n roots, and no more; it may always be decomposed into the product of n simple factors, and into the product of some simple and some quadratic factors which shall be aways real; some of its roots may be equal, and thus it may not have n *different* roots.

Let $\phi(x)=0$ be the equation, and let $x=\alpha_1$ be a root: divide $\phi(x)$ by $x-\alpha_1$, continuing the division until the remainder R_1 is free from x, and therefore only a function of α: let $\phi_1(x)$ be the quotient, and suppose $x=\alpha_2$ a root of the equation $\phi_1(x)=0$: divide $\phi_1(x)$ by $x-\alpha_2$ until we arrive at a remainder R_2 free from n, let $\phi_2(x)$ be the quotient, which is of $n-2$ dimensions, and continue the process until we come to a function of the first degree, which may be represented by $\phi_{n-1}(x)$ or $x-\alpha_n$. Hence

$$\phi(x)=(x-\alpha_1)\,\phi_1(x)+R_1$$
$$\phi_1(x)=(x-\alpha_2)\,\phi_2(x)+R_2$$
$$\phi_2(x)=(x-\alpha_3)\,\phi_3(x)+R_3$$
$$\cdots\cdots\cdots\cdots$$
$$\phi_{n-1}(x)=(x-\alpha_n)+R_n.$$

Put $x=\alpha_1$ in the first identity, $x=\alpha_2$ in the second, &c.; and since these quantities are the roots of $\phi(x)=0$, $\phi_1(x)=0$, &c., they make these functions vanish, therefore $R_1=0$, $R_2=0$, $R_3=0\ldots.R_n=0$; multiply together the left and right members of these equations thus reduced, and divide the product by $\phi_1(x).\phi_2(x).\phi_3(x)\ldots.\phi_{n-1}(x)$, and we get

$$\phi(x)=(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots\ldots(x-\alpha_n),$$

which quantity vanishes when $x=\alpha_1,\ \alpha_2\ldots\alpha_n$; and as no other quantity will make any simple factor vanish, and therefore would not make the product $\phi(x)$ vanish, it follows that the equation $\phi(x)=0$ has no more than these n roots, some of which however may be equal, so that $\phi(x)=0$ may have less than n different roots; but it can never have less than n simple factors.

Thus, $(x-a)^3=0$ is only true when $x=a$, it has no other root; it has not therefore 3 different roots, but it is the product of three simple factors $(x-a)(x-a)(x-a)$. (*Vide* Prop. XIII.)

Again, if α_p should be imaginary, suppose it of the form $\beta_1+\gamma_1\sqrt{-1}$, then there must be another root α_m as we have seen, of the form $\beta_1-\gamma_1\sqrt{-1}$, hence

$$(x-\alpha_p)(x-\alpha_m)=\{(x-\beta_1)-\gamma_1\sqrt{-1}\}.\{(x-\beta_1)+\gamma_1\sqrt{-1}\}$$
$$=(x-\beta_1)^2+\gamma_1^2,$$

which quadratic quantity being the sum of two squares, is essentially positive: by thus coupling the factors containing corresponding imaginary roots, it is obvious that $\phi(x)$ will be reduced to the sum

$$\phi(x)=(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots.(x-\alpha_{p-1})\{(x-\beta_1)^2+\gamma_1^2\}.$$
$$\{(x-\beta_2)^2+\gamma_2^2\}.\ldots.$$

and therefore $\phi(x)$ is always the product of real factors, simple or quadratic; and the species of imaginary quantity which may enter the solution of a quadratic is the only one which can enter the solution of an equation of any higher order.

19. The roots of an equation $\phi(x)=0$ being either real or imaginary, our next step is to consider some method for discovering the number of real roots and the number of imaginary; for this purpose we give

Proposition XVI.

Sturm's Theorem.

Let $\phi(x)$ be a rational and integer function of x, of which $\phi'(x)$ is the derived; divide $\phi(x)$ by $\phi'(x)$, in the manner of finding the greatest common measure, only with the peculiarity of always changing the sign of the remainder before making it a divisor, and continue this division until it terminates, either by giving a final remainder independent of x, or by a divisor which exactly measures the preceding.

These successive divisors after $\phi'(x)$ may be represented by $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$. Let A, B be any two numbers, of which A is the least (regard being had to its sign).

Substitute A for x in the series of functions following

$$\phi(x),\ \phi'(x),\ \phi_1(x),\ \phi_2(x),\ldots.$$

and register in the same order the signs of the results.

Count the number of sequences of two terms having contrary signs in this series of results; suppose a that number, substitute B now for x in the same series of functions, counting as before the number of variations of signs in consequent terms, suppose it $=b$.

There must exist $a-b$ real roots of the equation $\phi(x)=0$, which are greater than A and less than B, and there can exist no other real roots between these limits.

The process employed furnishes us with the following system of equations:

$$\begin{aligned}
\phi\ (x) &= Q_1 . \phi'(x) - \phi_1(x) \\
\phi'(x) &= Q_2 . \phi_1(x) - \phi_2(x) \\
\phi_1(x) &= Q_3\ \phi_2(x) - \phi_3(x) \\
\phi_2(x) &= Q_4\ \phi_3(x) - \phi_4(x) \\
&\text{\&c.}
\end{aligned}$$

First, we shall suppose $\phi(x)=0$ not to have equal roots, or, which is the same, that the function $\phi(x)$ has no equal simple factors.

Let α be a real root of the above equation, and suppose $\alpha-h$, $\alpha+h$, put for x in the series of functions

$$\phi(x) \quad \phi'(x) \quad \phi_1(x) \quad Q_2(x) \text{ \&c.}$$

They become for $\alpha-h$

$$\phi\ (\alpha)-\phi'\ (\alpha).h+\phi''\ (\alpha).\frac{h^2}{1.2}\text{ \&c.}$$

$$\phi'(\alpha)-\phi''(\alpha).h+\phi'''(\alpha).\frac{h^3}{1.2}\text{ \&c.}$$

$$\phi_1(\alpha)-\phi_1\ (\alpha).h+\phi''_1(\alpha).\frac{h^3}{1.2}\text{ \&c.}$$

&c., &c.,

in each of which we may suppose h so small that the sign of the whole will depend on that of its first term.

Their signs arranged in order are therefore the same as those of

$$-\phi'(\alpha),\ +\phi'(\alpha),\ \phi_1(\alpha),\ \phi_2(\alpha)\text{ \&c.}\quad \text{Since } \phi(\alpha)=0$$

in like manner if $\alpha+h$ be put for x, the signs of the results are the same as those of

$$\phi'(\alpha),\ \phi'(\alpha),\ \phi_1(\alpha),\ \phi_2(\alpha)\text{ \&c.}$$

Therefore, by varying x from a quantity a little below a real root to another above it, one variation of sign (to the left) is lost, that is exchanged for a permanence.

We have here supposed that none of the quantities $\phi_1(\alpha)$, $\phi_2(\alpha)$, &c. vanishes, but if one of them did, this would not alter the number of variations of signs.

Suppose $\phi_n(\alpha)$ to vanish, then since

$$\phi_{n-1}(\alpha)=Q_{n+1}\,\phi_n(\alpha)-\phi_{n+1}(\alpha)$$

we have $\phi_{n-1}(\alpha)=-\phi_{(n+1)}(\alpha)$, neither of which vanish, for if two such consecutive functions vanished, then tracing backwards by the equation $\phi_{n-2}(\alpha)=Q_n\,\phi_{n-1}(\alpha)=\phi^n(\alpha)$, we should have successively $\phi_{n-2}(\alpha)=0$, $\phi_{n-3}(\alpha)=0$), &c., up to $\phi'(\alpha)=0$, which is contrary to the hypothesis we have made. Hence $\phi_{n-1}(\alpha)$, $\phi_{n+1}(\alpha)$ form a variation.

Put now $\alpha-h$ for α: when h is small, $\phi_{n-1}(\alpha)$, $\phi_{n+1}(\alpha)$ will have the same signs as before, which were contrary, and, therefore, whatever sign $\phi_n(\alpha)$ may have, the three functions $\phi_{n-1}(\alpha-h)$, $\phi_n(\alpha-h)$, $\phi_{n+1}(\alpha-h)$ must either form first, a permanence, and then a variation, or first, a

variation, and then a permanence, in either case but one variation: thus no variation is lost by the vanishing of $\phi_n(\alpha)$.

If therefore x be increased by insensible degrees from $x=A$ to $x=B$, every time its value becomes that of a real root of the equation $\phi(x)=0$, the series of signs of $\phi(x)\ \phi'(x)\ \phi_1(x)\ \phi_2(x)\dots$ loses a variation and only then; hence there are as many roots between A and B as there are more variations for A than for B.

Suppose now there exist m equal roots for the equation $\phi(x)=0$, then $\phi(x)$ has m equal simple factors, or $\phi(x)=(x-a)^m.\ F(x)$, the latter function as well as the former being rational and integer, put $x+h$ for x

$$\text{hence } \phi(x+h)=(x-a+h)^m.\ F(x+h)$$

or

$$\phi(x)+\phi'(x).h, \text{ \&c.}$$
$$=\{(x-a)^m+m\,(x-a)^{m-1}\,h, \text{ \&c.}\}.\ \{F(x)+F'(x).h+\text{\&c.}\}$$

and equating the coefficients of h at both sides we have

$$\phi'(x)=(x-a)^{m-1}\{(x-a).\ F'(x)+m\,F(x)\}$$

that is, if an equation have m equal roots its derived will have $m-1$ of them, as we have seen before in Prop. XII. Now, since in this case $\phi'(x)$ and $\phi(x)$ have a common measure $(x-a)^{m-1}$, if we divide $\phi(x)$ by it, we reduce it to the case of unequal roots, and the same reasoning equally applies if other sets of roots are equal.

The same method as before applies in the case of equal roots, a, for in the expansions of $\phi(a-h)$, $\phi'(a-h)$, the first terms which do not vanish have necessarily contrary signs, as is obvious by inspection of their developments, and they have similar signs when $a+h$ is put for x, the series of quantities $\phi(x)$, $\phi'(x)$, $\phi_1(x)$, $\phi_2(x)$,....$\phi_{n-m}(x)$, are $n-m+2$ in number, and the number of *different* roots is $n-m+1$, which is the greatest number of variations of signs in this series; therefore the number of *different* roots between A and B, is the excess of the number of variations of signs when the less quantity A is substituted for x above that arising when B is substituted.

It is easily seen that the same reasoning would apply if there existed different sets of equal roots.

Corollary. From hence follows an easy method of finding the whole number of real and of imaginary roots in an equation, when the number of the quantities $\phi(x)$, $\phi'(x)$, $\phi_1(x)$, &c. is $n+1$, which it will generally be, n being the dimensions of $\phi(x)$.

When $+(\infty)$, or an exceedingly great positive number, is put for x in this series of quantities, the signs of the results will necessarily be the same as those of the first terms by Prop. I. Let m be the number of variations of signs in consecutive terms of this series.

When $-(\infty)$ is put for x, the signs of those functions which are of even dimensions will be the same as before, but those of odd dimensions will be the contrary.

There will, in the latter substitution, be therefore as many variations of consequent terms as there were permanences in the former, namely, $n-m$.

And since all the real roots are comprised between these limits, their number by this Proposition must be $(n-m)-m$, or $n-2m$.

Hence, the number of impossible roots is $2m$; there exist therefore

as many pairs of impossible roots as there are variations in the signs of the first terms of the functions $\phi(x)$, $\phi'(x)$, $\phi_1(x)$, $\phi_2(x)$, &c.

Example. Let $\phi(x)=x^3-6x^2+8x+40$
$\phi'(x)=3x^2-12x+8$

Multiply $\phi(x)$ by 3 to avoid fractions, and in like manner the subsequent dividends or divisors may be multiplied by any *positive* number.

$$\begin{array}{r l}
3x^2-12x+8) & 3x^3-18x^2+24x+120\ (x-2 \\
 & \underline{3x^3-12x^2+\ 8x\qquad\quad} \\
 & \quad -6x^2+16x+120 \\
 & \quad \underline{-6x^2+24x-\ 16} \\
 & \qquad\qquad -8x+136
\end{array}$$

divide by -8, which will change the sign, and we have

$$\phi_1(x)=x-17$$

$$\begin{array}{r l}
x-17) & 3x^2-12x+8\ (3x+39 \\
 & \underline{3x^2-51x} \\
 & \quad 39x+8 \\
 & \quad \underline{39x-663} \\
 & \qquad\quad 671
\end{array}$$

$$\phi_2(x)=-671$$

The series of signs of the first terms of $\phi(x)$, $\phi'(x)$, $\phi_1(x)$, $\phi_2(x)$, is in this case $+\ +\ +\ -$; there being one variation, the equation must have one pair of impossible roots, and therefore only one real root.

Example 2. $\phi(x)=x^3-3px+2q$
$\phi'(x)=3x^2-3p$

$$\begin{array}{r l}
x^2-p) & x^3-3px+2q\ (x \\
 & \underline{x^3-px\qquad\quad} \\
 & \quad -2px+2q
\end{array}$$

$$\phi_1(x)=px-q$$

$$\begin{array}{r l}
px-q) & x^2-p\ \left(\frac{x}{p}+\frac{q}{p^2}\right. \\
 & \underline{x^2-\frac{qx}{p}} \\
 & \frac{q}{p}.x-p \\
 & \underline{\frac{q}{p}.x-\frac{q^2}{p^2}} \\
 & \frac{q^2}{p^2}-p
\end{array}$$

$\phi_2(x)=p^3-q^2$ for p^2 is necessarily positive.

The series of signs of the first terms are the same as those of $1,\ 1,\ p,\ p^3-q^2$

If p be negative there is one variation, and therefore only one real root; if p be positive and $p^3<q^2$ the same thing happens; if $p^3>q^2$ there are no impossible roots.

20. Another theorem for finding the number of real roots of a given equation, which lie between two assigned numbers, was given by

Fourier; its application is very easy, but it has the great fault of only indicating a number which the sought number of real roots *does not exceed.* The discovery of Sturm renders almost useless the tedious process by which Fourier sought to perfect his original proposition, which is as follows:—

Proposition XVII.

Fourier's Theorem.

Let two quantities, a, b, of which a is the least, be substituted for x in the functions $\phi(x)$, $\phi'(x)$, $\phi''(x)$, $\phi'''(x)$, &c., each of which is the derived of the preceding, and let the signs of the results be noted; there cannot be more real roots of the equation $\phi(x)=0$ lying between a and b than the excess of the number of alternations of signs which result from the substitution of a, over the number resulting from the substitution of b.

Suppose α to be a root of the given equation, and h a very small quantity, and let $\alpha-h$ be put for x in the above series of functions, the results are

$$-\phi'(a).h+\&c.,\ \phi'(a)+\&c.,\ \phi''(a)+\&c.,\ \phi'''(a)+\&c.$$

Now let $-h$ be changed to $+h$, which is the same as putting $a+h$ for x, the result will be

$$+\phi'(a).h+\&c.,\ \phi'(a)+\&c.,\ \phi''(a)+\&c.,\ \phi'''(a)+\&c.$$

The two first terms in the former series give an alternation, and in the latter a permanence of signs; and supposing none of the derived functions to vanish when $x=a$, we see that an alternation of signs is lost by passing from a quantity a little below a root to one a little above.

But this series may also be affected by the vanishing of any of the derived functions (being continuous, they cannot change signs without passing through zero). Suppose β makes $\phi'''^{(m)}(x)$ vanish when substituted for x, and taking h very small, this part of the series of functions, when $\beta-h$ and $\beta+h$ are respectively put for x, will be

$$\phi''^{(m-1)}(\beta)-\&c.,\ -\phi'''^{(m+1)}(\beta).h+\&c.,\ \phi'''^{(m+1)}(\beta)-\&c.$$
$$\phi''^{(m+1)}(\beta)+\&c.,\ +\phi'''^{(m+1)}(\beta).h+\&c.,\ \phi'''^{(m+1)}(\beta)+\&c.$$

Now if $\phi''^{(m-1)}(\beta)$, $\phi''^{(m+1)}(\beta)$ have contrary signs, the first series will give an alternation followed by a permanence, the second a permanence followed by an alternation, the total number of alternations then is unaltered.

But if $\phi''^{(m-1)}(\beta)$, $\phi'''^{(m+1)}(\beta)$ have like signs, the first series gives two alternations, the second two permanences; in this case two alternations have disappeared.

And if several consecutive derived functions vanish for any value of x, then it may be shown in the same manner that an even number of alternations disappear.

By increasing therefore insensibly the value of x from a to b, an alternation is lost as often as we meet with a root of the given equation in that interval, *beside which* an even number of alternations *may* disappear when we meet with a root of any of the derived equations.

The following example is taken from Fourier:

Given $\phi(x)=x^5-3x^4-24x^3+95x^2-46x-101=0.$

$\phi'(x)=5x^4-12x^3-72x^2+190x-46.$

$\phi''(x)=20x^3-36x^2-144x+190.$

$\phi'''(x)=60x^2-72x-144.$

$\phi^{\text{iv}}(x)=120x-72.$

$\phi^{\text{v}}(x)=120.$

$x=-10$	Number of alternations of signs	$=5$
$x=-1$	. . . . . .	$=4$
$x=0$	. . . . . .	$=3$
$x=1$	. . . . . .	$=3$
$x=10$	. . . . . .	$=0$

Hence all the roots are between -10 and $+10$; there is one real root between -10 and -1, another between -1 and 0; no root exists between 0 and 1, and one at least must exist between 1 and 10; these substitutions leave us in uncertainty with respect to the other two roots, whether they also lie between 1 and 10, or are imaginary; Sturm's theorem is free from this capital defect.

21. The rule given by the early analysts, Descartes, Harriot, &c., relative to the number of positive and negative roots, and which is proved in the *Theory of Algebraical Expressions**, p. 14, namely, that the number of positive roots cannot exceed the number of alternations, nor can the number of negative roots exceed the number of permanences of the signs of the consecutive terms of the equation it contained in the preceding theorem of Fourier.

For when 0 is put for x, the signs of $\phi(x)$, $\phi'(x)$, $\phi''(x)$, &c. are evidently the same as those of the terms of the equation from right to left.

And when x is supposed exceedingly great and positive, the signs of the same functions are then all positive.

Therefore there cannot be more positive roots than there are alternations of signs. (Prop. XVI.)

Similarly, when x is supposed very great and negative, the series of signs of $\phi(x)$, $\phi'(x)$, $\phi''(x)$, &c., form only alternations, and their number is evidently the sum of the number of alternations and permanences in the terms of $\phi(x)$, and therefore exceeds the number of alternations for $x=0$ by the number of permanences in the terms of the equation; hence the number of negative roots cannot exceed this.

The same theorem comprehends another rule, given by Newton, for finding a superior limit to the roots of an equation, that is, a number greater than the greatest real root; it is this: put $x=y+e$, and assign to e, by trial, a value so great as to render positive all the terms of the transformed equation arranged according to the powers of y; that value is the superior limit.

For by Prop. (3), the transformed equation deduced from $\phi(x)=0$ will be

$$\phi(e)+\phi'(e).y+\frac{\phi''(e)}{1.2}.y^2+\frac{\phi'''(e)}{1.2.3}.y^3+\text{\&c.}=0;$$

and since $\phi(e)$, $\phi'(e)$, $\phi''(e)$, &c. are all positive for the assigned value of e, no root of the equation $\phi(e)=0$, or $\phi(x)=0$, can lie between this value and positive infinity.

* Library of Useful Knowledge.

22. There are some remarkable relations existing between the roots of an equation and those of the derived equation, which, though not always useful for finding numbers between which the roots of the primitive equation lie, have considerable use in the theory of equations.

Proposition XVIII.

If the real roots of an equation $\phi(x)=0$ be substituted in the order of magnitude, beginning with the greatest, in the derived function $\phi'(x)$, they will produce alternately positive and negative results; or, in the case of equal roots, they will make this function vanish.

Suppose $a, b, c,$ &c. to be the real roots, decreasing in magnitude, of the equation $\phi(x)=0$; and suppose $\alpha+\beta\sqrt{-1}$, $\alpha-\beta\sqrt{-1}$, to be a pair of imaginary roots of the same, if any such exist.

From the real roots are formed real simple factors of $\phi(x)$, viz., $x-a$, $x-b$, $x-c$, &c.

From the imaginary roots real quadratic factors are formed, such as $(x-\alpha)^2+\beta^2$. (Prop. 14.) These factors are essentially positive when any real value is assigned to x; the sign of $\phi(x)$ is therefore the same as that of the product of its real simple factors.

Now $\phi(x)$ is of the form $(x-a).\mathrm{P}$, where P denotes the product of the remaining simple and quadratic factors, its sign depending only on the former.

Let $\phi'(x)$ be the function derived from $\phi(x)$, and P′ from P, we easily find $\phi'(x)=(x-a).\mathrm{P}'+\mathrm{P}$.

The simple factors of P, viz. $x-b$, $x-c$, &c. are all positive when a is substituted for x, and by the last equation $\phi'(x)$ is also positive.

Again, we may put $\phi(x)=(x-b).\mathrm{Q}$, and therefore $\phi'(x)=(x-b).\mathrm{Q}'+\mathrm{Q}$; and of the factors of Q, viz. $x-a$, $x-c$, &c. one only is negative when b is put for x, namely $b-a$; the substitution of b renders therefore $\phi'(x)$ negative.

In the same manner it will easily be perceived that the substitution of c for x would give to $\phi'(x)$ a positive value, and so on.

From this proposition we see that between two consecutive roots of the primitive equation $\phi(x)=0$, an odd number of roots (one at least) of the derived equation must exist.

Also, that the first derived equation has at least as many real roots as the primitive, wanting one; and the mth successive derived function has at least as many real roots as the primitive wanting m, all which lie between the greatest and least of the primitive.

When there exist p equal roots in the given equation, it is of the form $(x-a)^p.\mathrm{R}$; and therefore the derived is of the form $(x-a)^{p-1}.\mathrm{S}$, that is, $p-1$ of such roots descend to the first derived, and similarly $p-2$ to the second, and so on, and the same law holds if there are several systems of equal roots.

If all the roots of an equation be real, all the roots of any of its derived equations will also be real.

The converse theorems to those above given are not necessarily true.

Example. $\phi(x)=x^n(1-x)^n$.

The equation $\phi(x)=0$ has n roots each equal to zero, and other n roots each equal to unity.

The nth derived equation is

$$0=1-\frac{n}{1}.\frac{n+1}{1}.x+\frac{n(n-1)}{1.2}.\frac{(n+1)(n+2)}{1.2}.x^2-, \text{ \&c.}$$

This equation must have all its roots real, and between 0 and 1.

23. It is frequently convenient to make equations undergo transformations, which, as they in general affect all the roots in the same manner, will produce, as coefficients to the terms of the transformed equations, *symmetrical* functions of these roots; an abridged but expressive notation to indicate the symmetrical function which enters the process, with a little attentive practice, has the desirable effect of removing a mass of unnecessary labour, and of giving with distinctness, at a glance, the mutual relations of such functions. Perhaps the simplest mode of representing symmetrical functions is by prefixing the sign Σ before one of the terms of the symmetrical function, which is taken as a type of all the others, and from which they may be generated by merely changing in all possible ways so as to produce *different* combinations, the roots which enter that term, the prefixed Σ denoting the sum of all the *similar* but *different* terms thus generated, which sum is the symmetrical function to be expressed.

Examples for Practice.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots.\alpha_n$, be the n roots of an equation of n dimensions.

Let $S_1, S_2, S_3, \ldots.S_n$, be respectively the sums of their first, second, third, &c. powers.

Let $a_1, a_2, a_3, \ldots.a_n$, be the sums respectively of the roots themselves, of their combinations, two and two, three and three, &c.

First example. $S_1=\Sigma\alpha_1$. $a_1=\Sigma\alpha_1$.
therefore $S_1-a_1=0$.

Second example $S_2=\Sigma\alpha_1{}^2$
$a_1=\Sigma\alpha_1$ $S_1=\Sigma\alpha_1$

but $a_1.S_1$ is not the same as $\Sigma\alpha_1{}^2$, it includes besides terms of the form $\alpha_1\alpha_2$, and each term not altering by permutation must enter twice;

$$\text{hence } a_1S_1=\Sigma\alpha_1{}^2+2\Sigma\alpha_1\alpha_2;$$
and since $2a_2=2\Sigma\alpha_1\alpha_2$, we have
$$S_2-a_1S_1+2a_2=0.$$

Third example. $S_3=\Sigma\alpha_1{}^3$.
$$-a_1S_2=-\Sigma\alpha_1.\Sigma\alpha_1{}^2=-\Sigma\alpha_1{}^3-\Sigma\alpha_1{}^2\alpha_2.$$
$$a_2S_1=\Sigma\alpha_1\alpha_2.\Sigma\alpha_1=\Sigma\alpha_1{}^2\alpha^2+3\Sigma\alpha_1\alpha_2\alpha_3,$$

because the change of roots in the latter term produces necessarily three like terms;

and since $-3a_3=-3\Sigma\alpha_1\alpha_2\alpha_3$,

we have by addition $S_3-a_1S_2+a_2S_1-3a_3=0$.

Fourth example; m not greater than n.

$$S_m=\Sigma\alpha_1{}^m.$$
$$-a_1S_{m-1}=-\Sigma\alpha_1.\Sigma\alpha_1{}^{m-1}=-\Sigma\alpha_1{}^m-\Sigma\alpha_1\alpha_2{}^{m-1}$$
$$a_2S_{m-2}=\Sigma\alpha_1\alpha_2\Sigma\alpha_1{}^{m-2}=\Sigma\alpha_1{}^{m-1}\alpha_2+\Sigma\alpha_1\alpha_2\alpha_3{}^{m-2}.$$
$$-a_3S_{m-3}=-\Sigma\alpha_1\alpha_2\alpha_3\Sigma\alpha_1{}^{m-3}=-\Sigma\alpha_1{}^{m-2}\alpha_2\alpha_3-\Sigma\alpha_1\alpha_2\alpha_3\alpha_4{}^{m-3}.$$

in each of which the last term of the right-hand member is the same, with a contrary sign to the symmetrical function which is the first term in the same member of the next identity; but as the index diminishes in these combinations to $m-1$, $m-2$, &c., successively, it will be reduced to unity, in the identity of which the left member is $\pm a_{m-1} S_1$, it will not then be altered by putting successively all the roots for that with the index so diminished, and as the sign Σ implies that only *different* combinations enter under it, we must give this term a coefficient m, or

$$a_{m-1} S_1 = \Sigma a_1^2 a_2 a_3 \dots a_{m-1} + m \Sigma a_1 a_2 a_3 \dots a_m ;$$

and by the addition of these identities we have

$$S_m - a_1 S_{m-1} + a_2 S_{m-2} - a_3 S_{m-3} \dots \pm a_{m-1} S_1 \mp m a_m = 0,$$

which is a verification, by actual process, of the theorem of Newton for the sums of the powers of the roots, which has been proved directly in the *Algebraical Expressions*, p. 18, by Mr. Drinkwater.

When $m > n$, the index under Σ in the last term of the right hand members never reduces itself to unity in all the equations where both the factors multiplied are polynomials; it is for this reason that the theorem then becomes

$$S_m - a_1 S_{m-1} + a_2 S_{m-2} - a_3 S_{m-3} + \dots \pm a_n S_{m-n} = 0.$$

These forms of Newton's theorem have been long used, first to find the sums of the powers of the roots when the coefficients of an equation $x^n - a_1 x^{n-1} + a_2 x^{n-2} - \dots \pm a_n = 0$, are given, and, secondly, to find the coefficients when the sums of the powers are given, by both which processes combined the transformations of algebraic equations can be effected with scarcely any difficulty but the length of the process.

24. The sums of the powers of the roots thus found are implicit, that is, expressed not explicitly in terms of the coefficients, but made to depend on each other by a formula of reduction; they may be obtained explicitly by the following theorem.

Proposition XIX.

Divide the left hand member of an arranged algebraical equation by its first term, or that which has the greatest exponent, the equation becoming then of the form $1 + P = 0$, in which P contains only negative powers of x. Take next the logarithm of this quotient by the formula,

$$\text{Log.}\,(1+P) = P - \frac{1}{2}.P^2 + \frac{1}{3}P^3 \dots \pm \frac{1}{m}P^m, \text{ \&c.}$$

and commencing with the last-written term select the coefficient of x^{-m} in that and as many of the preceding as contain it; this quantity, when multiplied by m, and the sign changed, will be the sum of the mth powers of the roots.

But if the equation be divided by the last instead of the first term of its left-hand member, and then the logarithm taken, &c. as before, we should thus obtain the sum of the inverse mth powers of the roots, by taking the coefficient of x^m and multiplying it by $-m$.

For let $a_1, a_2, a_3, \dots \dots a_n$ be the roots of the equation; decompose the left member of the equation into simple factors, it is equivalent to

$$(x-a_1)\ (x-a_2)\ (x-a_3)\ldots\ldots(x-a_n).$$

Hence, dividing by x^n,

$$1+\mathrm{P}=\left(1-\frac{a_1}{x}\right)\left(1-\frac{a_2}{x}\right)\left(1-\frac{a_3}{x}\right)\ldots\ldots\left(1-\frac{a_n}{x}\right);$$

therefore

$$\text{Log.}\ (1+\mathrm{P})=\text{Log.}\left(1-\frac{a_1}{x}\right)+\text{Log.}\left(1-\frac{a_2}{x}\right)+\text{Log.}\left(1-\frac{a_3}{x}\right)+$$
$$\ldots\ldots\ldots+\text{Log.}\left(1-\frac{a_n}{x}\right)$$
$$=-\frac{a_1}{x}-\frac{1}{2}\cdot\frac{a_1^2}{x^2}-\frac{1}{3}\cdot\frac{a_1^3}{x_3}-\ldots\ldots-\frac{1}{m}\cdot\frac{a_1^m}{x^m}-,\ \&c.$$
$$-\frac{a_2}{x}-\frac{1}{2}\cdot\frac{a_2^2}{x^2}-\frac{1}{3}\cdot\frac{a_2^3}{x^3}-\ldots\ldots-\frac{1}{m}\cdot\frac{a_2^m}{x^m}-,\ \&c.$$
$$\cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot$$
$$-\frac{a_n^1}{x}-\frac{1}{2}\cdot\frac{a_n^2}{x^2}-\frac{1}{3}\cdot\frac{a_n^3}{x^3}-\ldots\ldots-\frac{1}{m}\cdot\frac{a_n^m}{x^m}-,\ \&c.$$
$$=-\left\{\frac{\mathrm{S}_1}{x}+\frac{1}{2}\cdot\frac{\mathrm{S}_2}{x^2}+\frac{1}{3}\cdot\frac{\mathrm{S}_3}{x^3}+\ldots\ldots+\frac{1}{m}\cdot\frac{\mathrm{S}^m}{x^m}+,\ \&c.\right\}$$

From whence it follows that $-\frac{\mathrm{S}_m}{m}$ is the coefficient of x^{-m} in Log. $(1+\mathrm{P})$.

In like manner, suppose the equation divided by its last term, which is the same as $-a_1.a_2.a_3\ldots.a_n$, and the quotient resulting from the left-hand member to $1+\mathrm{Q}$, we have

$$1+\mathrm{Q}=\left(1-\frac{x}{a_1}\right)\left(1-\frac{x}{a_2}\right)\left(1-\frac{x}{a_3}\right)\ldots\ldots\left(1-\frac{x}{a_n}\right).$$

Put now $\mathrm{S}_{-1}=\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots\ldots+\frac{1}{a_n}$.

$$\mathrm{S}_{-2}=\frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2}+\ldots\ldots+\frac{1}{a_n^2}.$$
$$\cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot$$
$$\mathrm{S}_{-m}=\frac{1}{a_1^m}+\frac{1}{a_2^m}+\frac{1}{a_3^m}+\ldots\ldots+\frac{1}{a_n^m}.$$

Hence $\text{Log.}\ (1+\mathrm{Q})=-\Big\{\mathrm{S}_{-1}.x+\frac{1}{2}.\mathrm{S}_{-2}.x^2+\frac{1}{3}.\mathrm{S}_{-3}\,x^3+\ldots\ldots\ldots$
$$+\frac{1}{m}.\ \mathrm{S}_{-m}\,x^m+\&c.\Big\},$$

by which the second part of the proposition is manifest.

Example 1.—To find the sum of the direct and inverse mth powers of the roots of a quadratic equation, $x^2+ax+b=0$.

By the proposition just proved we have

$$-\frac{1}{m}.\ \mathrm{S}_m = \text{coefficient of } x^{-m} \text{ in Log.}\left\{1+\left(\frac{a}{x}+\frac{b}{x^2}\right)\right\}.$$

Rejecting the terms which would involve negative powers after x^{-m}, and inverting the order of the terms; lastly, multiplying by $-m$, we shall have

$$S_m = \text{coefficient of } x^{-m} \text{ in } (-1)^m\left\{\left(\frac{a}{x}+\frac{b}{x^2}\right)^m - \frac{m}{m-1}\left(\frac{a}{x}+\frac{b}{x^2}\right)^{m-1} + \frac{m}{m-2}\left(\frac{a}{x}+\frac{b}{x^2}\right)^{m-2} - \&c.\right\}$$

$$= (-1)^m\left\{a^m - m.a^{m-2}b + \frac{m(m-3)}{1.2}.a^{m-4}b^2 - \frac{m(m-4)(m-5)}{1.2.3}.a^{m-6}b^3 + \&c.\right\}$$

the number of terms being $\frac{m}{2}+1$, or $\frac{m+1}{2}$, according as m is an even or odd integer.

Similarly,

$$-\frac{1}{m}S_{-m} = \text{coefficient of } x^m \text{ in Log.}\left\{1+\left(\frac{a}{b}.x+\frac{1}{b}.x^2\right)\right\}.$$

Therefore, S_{-m} = coefficient of x^m in $(-1)^m\left\{\left(\frac{a}{b}.x+\frac{1}{b}.x^2\right)^m - \frac{m}{m-1}\left(\frac{a}{b}.x+\frac{1}{b}.x^2\right)^{m-1} + \&c.\right\}$.

$=\frac{S_m}{b^m}$, which result is verified by recollecting that b^m is the product of the mth powers of the same two quantities, for which S_m is the sum of the mth powers.

Example 2.—To find the sum of the inverse mth powers of the roots of the equation $x^n - ax + b = 0$.

Here we have

$$-\frac{1}{m}.S_m = \text{coefficient of } x^m \text{ in Log.}\left\{1-\frac{1}{b}(ax-x^n)\right\}.$$

When the logarithm is expanded the only terms which *may* contain x^m are

$$-\frac{1}{b}(ax-x^n)-\frac{1}{2b^2}(ax-x^n)^2-\frac{1}{3b^3}(ax-x^n)^3\ldots\ldots-\frac{1}{mb^m}(ax-x^n)^m.$$

Therefore

$$S_{-m} = \text{coefficient of } x^m \text{ in } \frac{1}{bm}\left\{(ax-x^n)^m + \frac{m}{m-1}.b(ax-x^n)^{m-1} + \frac{m}{m-2}.b^2(ax-x^n)^{m-2}\ \&c.\right\}$$

The general term of the series between the brackets is—

$$\frac{m}{m-k}.b^k(ax-x^n)^{m-k}.$$

Now, $(ax-x^n)^{m-k} = a^{m-k}x^{m-k} - (m-k)a^{m-k-1}x^{m+n-k-1} + \frac{(m-k)(m-k-1)}{1\ .\ 2}a^{m-k-2}x^{m+2n-k-2}$ &c.,

the only values of k which would allow the existence of a term containing x^m on this expansion are obviously

$$k=0,\ n-1,\ 2n-2,\ 3n-3, \text{ \&c.}; \text{ hence}$$

$$S_{-m}=\text{coefficient of } x^m \text{ in } \frac{1}{b^m}\Big\{(ax-x^n)^m+\frac{m}{m-n+1}.b^{n-1}(ax-x^n)^{m-n+1}$$

$$+\frac{m}{m-2n+2}b^{2n-2}(ax-x^n)^{m-2n+2}+\text{\&c.}\Big\}$$

$$=\frac{1}{b^m}\Big\{a^m-ma^{m-n}b^{n-1}+\frac{m(m-2n+1)}{1\ .\ 2}a^{m-2n}b^{2n-2}$$

$$-\frac{m(m-3n+2)(m-3n+1)}{1\ .\ 2\ .\ 3}a^{m-3n}b^{3n-3}+\text{\&c.}\Big\}$$

the number of terms being the integer next greater than $\frac{m}{n}$.

As a particular instance suppose $a=0$, $b=-1$.

When m is not a multiple of n the whole series vanishes, and when it is a multiple the required sum is equal to n; the equation in this case is $x^n-1=0$, and the same property of the roots of unity is equally true for the sums of their positive powers; for if we put $x=\frac{1}{y}$, the equation becomes $y^n-1=0$, the roots of which are necessarily the reciprocals of the roots of the former, which, being the same in form, it follows that, amongst the roots of the equation $x^n-1=0$, each has its reciprocal, and therefore the sum of the direct and inverse mth powers must be alike.

Example 3.—To find the sum of the inverse mth powers of the roots of the equation $x^n-ax^p+1=0$, where $n>p$, and prime to p.

Expanding the logarithm we have—

$$\frac{1}{m}.S_{-m}=\text{coefficient of } x^m \text{ in } +(ax^p-x^n)+\frac{1}{2}(ax^p-x^n)^2$$

$$+\frac{1}{3}(ax^p-x^n)^3, \text{ \&c.}$$

Now $(ax^p-x^n)^k=a^k.x^{pk}-ka^{k-1}x^{kp+n-p}+\frac{k.(k-1)}{1\ .\ 2}.a^{k-2}x^{kp+2(n-p)}$, &c.

And in order that x^m may enter, it is necessary that some of the following equations may be possible in integers, viz.:—

$$kp=m,\ m+p-n,\ m+2p-2n,\ m+3p-3n, \text{ \&c.}$$

First suppose m to be divisible by p; and since p is prime to n it can measure no multiple of n between 0 and pn, between pn and $2pn$, &c.; hence the possible values of k are $\frac{m}{p}$, $\frac{m}{p}+p-n$, $\frac{m}{p}+2p-2n$, &c., negative terms being excluded; hence $\frac{1}{m}S_{-m}$ is the coefficient of x^m in the series

$$\frac{p}{m}(ax^p-x^n)^{\frac{m}{p}}+\frac{1}{\frac{m}{p}+p-n}(ax^p-x^n)^{\frac{m}{p}+p-n}+\frac{1}{\frac{m}{p}+2p-2n}$$

$$(ax^p-x^n)^{\frac{m}{p}+2p-2n}, \text{ \&c.}$$

or, $$S_{-m}=pa^{\frac{m}{p}}$$

$$+\frac{m.\left(\frac{m}{p}+p-n-1\right).\left(\frac{m}{p}+p-n-2\right)..\left(\frac{m}{p}-n+1\right)}{1\quad.\quad 2\quad.\quad 3\quad.\quad p}.a^{\frac{m}{p}-n}(-1)^{p}$$

$$+\frac{m.\left(\frac{m}{p}+2p-2n-1\right).\left(\frac{m}{p}+2p-2n-2\right)..\left(\frac{m}{p}-2n+1\right)}{1\quad.\quad 2\quad.\quad 3\quad.\quad 2p}.a^{\frac{m}{p}-2n}$$

+, &c.

When p does not measure m, put

$$pk=m+k'(p-n),\text{ or }pk+(n-p)k'=m,$$

and if K be the greatest value of k which satisfies this indeterminate equation, its other values are $K+p-n$, $K+2p-2n$, &c.; and then

$$\frac{1}{m}S_{-m}=\text{coefficient of }x^{m}\text{ in }\frac{1}{K}(ax^{p}-x^{n})^{K}+\frac{1}{K+p-n}(ax^{p}-x^{n})^{K+p-n}$$

+, &c.

And if κ be the least value of k, its other values are $\kappa+p$, $\kappa+2p$, &c., and these determine the *place of the term* which must be selected from each of the above binomials.

25. In the next place, it will be very useful to be able to find explicitly the coefficients of an equation when the sums of the powers of its roots are supposed to be known.

Proposition XX.

Form the expansions $1-S_1.h+\frac{S_1^2}{1.2}.h^2-\frac{S_1^3}{1.2.3}.h^3$, &c. to $n+1$ terms.

$1-\frac{S_2}{2}.h^2+\frac{S_2^2}{1.2.2^2}.h^4-\frac{S_2^3}{1.2.3.2^3}.h^6$, &c., the number of terms being the integer next above $\frac{n}{2}$.

$1-\frac{S_3}{3}.h^3+\frac{S_3^2}{1.2.3^2}.h^6-\frac{S_3^3}{1.2.3.3^3}.h^9$, &c., the number of terms being the integer next above $\frac{n}{3}$.

And so on to the final series, consisting only of two terms, viz.:—

$$1-\frac{S_n}{n}.h^n.$$

If S_1, S_2, S_3......S_n, be the sums of the first, second, third .., nth powers of the roots of the equation $x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n$, then a_m will be found by taking the coefficient of h^m in the product of the first m series given above.

For by the last Proposition

$$\log\left\{1+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots+\frac{a^n}{x^n}\right\}=-\frac{S_1}{x}-\frac{1}{2}.\frac{S_2}{x^2}-\frac{1}{3}.\frac{S_3}{x^3}\text{ \&c.}$$

write h instead of $\frac{1}{x}$, and denote by ε the base of Naperian logarithms.

Hence $1+a_1h+a_2h^2+\ldots+a_nh^n = \varepsilon^{-S_1h}.\varepsilon^{-\frac{1}{2}S_2h^2}.\varepsilon^{-\frac{1}{3}S_3h^3}\ldots$ &c.
and the series given in the present proposition are the expansions according to the powers of h of each of the factors in the right hand member of this identity; only the terms which have higher powers of h than the nth are rejected, as from the nature of the identity the total coefficient of any such power must be zero.

Example 1. To find a_1, a_2, a_3, &c.

$$a_1 = -S_1 \quad a_2 = \frac{S_1^2}{1.2} - \frac{S_2}{2} \quad a_3 = -\frac{S_1^3}{1.2.3} + S_1.\frac{S_2}{2}, \text{ \&c.}$$

Example 2. To form an equation such that its roots may possess the property $S_1=0$, $S_2=0$, $S_3=0\ldots S_{n-1}=0$, $S_n=c$.

In this case the first $n-1$ series contain no power of h; therefore

$a_1=0$, $a_2=0\ldots a_{n-1}=0$, $a_n = -\frac{S_n}{n} = -\frac{c}{n}$, the required equation

is therefore $x^n - \frac{c}{n} = 0$.

Example 3. Given $S_1=0$, $S_2=0\ldots S_p=p$, $S_{p+1}=0\ldots S_{n-1}=0$, $S_n=n$.

The two series to be multiplied in this case are

$$1-h^p+\frac{h^{2p}}{1.2}, \text{ \&c., and } 1-h^n, \text{ \&c.}$$

Suppose p does not measure n, and k the greatest number of times it is contained in n, the product of the two is

$$1-h^p+\frac{h^{2p}}{1.2}+\ldots+(-1)^k.\frac{h^{kp}}{1.2\ldots k}-h^n, \text{ \&c.}$$

the required equation is therefore

$$x^n-x^{n-p}+\frac{1}{1.2}.x^{n-2p}+\frac{1}{1.2.3}.x^{n-3p}\ldots\pm\frac{1}{1.2\ldots k}.x^{n-kp}-1=0.$$

But if p measures n, the two last terms must be united. In like manner may the coefficients be found when the sums of the negative powers of the roots are given.

26. Scholium. In the two preceding propositions we have supposed the reader acquainted with the expansions of a^x and $\log(1+x)$; if not, their investigations are here supplied.

To expand a^y according to the powers of y.

Let $a^x=f(x)$; then, since $a^x.a^y=a^{x+y}$, we have

$$f(x).f(y)=f(x+y)=f(x)+f'(x)y+f''(x).\frac{y^2}{1.2}+f'''(x).\frac{y^3}{1.2.3}+\text{\&c.}$$

therefore, $f(y)=1+\frac{f'(x)}{f(x)}.y+\frac{f''(x)}{f(x)}.\frac{y^2}{1.2}+\frac{f'''(x)}{f(x)}.\frac{y^3}{1.2.3}+$ &c.

where $f'(x)$, $f''(x)$, &c. are the successive derived functions of $f(x)$.

But it is clear that y being arbitrary, the expansion of $f(y)$, that

is a^y cannot contain x, and for the same reason $\frac{f'(x)}{f(x)}$ cannot contain y; therefore all the coefficients in the above series are independent both of x and y, and consequently must be some functions of a; let us therefore put $\frac{f'(x)}{f(x)}=\phi(a)$; hence we find

$$f'(x)=\phi(a).f(x);\ f''(x)=\phi(a).f'(x)=(\phi a)^2,\ f(x)$$
$$f'''(x)=(\phi a)^3.f(x),\ \&c.$$

by substituting these values we obtain

$$f(y)=1+\phi(a).y+\left(\phi(a)\right)^2.\frac{y^2}{1.2}+\frac{(\phi a)^3.y^3}{1.2.3}+\ \&c.\ =a^y.$$

Now, for some particular value of a, which we denote by ε, the value of $\phi(a)$ will be unity; this quantity ε is readily found by putting in the preceding expansion

$$\phi(a)=1,\ y=1,\ a=\varepsilon,$$

which gives $$1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\ \&c.=\varepsilon=2.712818,\ \&c.$$

Hence $$1+y+\frac{y^2}{1.2}+\frac{y^3}{1.2.3}+\ \&c.\ =\varepsilon^y;$$

write now $y\phi(a)$ for y in this expansion, and compare it with the former, and we have $\varepsilon^{y\phi(a)}=a^y$, therefore $\varepsilon^{\phi(a)}=a$; and hence, by the definition of a logarithm $\phi(a)$ is the log. of a to the base ε, that is the Naperian log. of a.

To expand the Naperian log. of $1+x$.

Let $1+x=a$, hence $(1+x)^y=1+\phi(a).y+(\phi a)^2.\frac{y^2}{1.2}+\ \&c.,$

where $\phi(a)$ is the Naperian log. of $1+x$, and is the coefficient of y in the expansion of $(1+x)^y$.

But also $$(1+x)^y=1+y.x+\frac{y(y-1)}{1.2}.x^2+\frac{y(y-1)(y-2)}{1.2.3}.x^3$$
$$+\frac{y(y-1)(y-2)(y-3)}{1.2.3.4}.x^4+\ \&c.$$

Select now the coefficient of the simple y from each term of this expansion, and it must be equal to the coefficient of the same quantity found by the former method. Hence

$$\text{Nap. log.}\,(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ \&c.$$

Our object here is only to prove those expansions which are of the most frequent application in analysis, and not to follow out the different properties of logarithmic series.

27. We shall now give some instances of simple transformations of equations, pointing out the uses to which they are subservient.

Problem I.

To increase or diminish the roots of an equation by a given quantity e.

Let $\phi(x)=0$ be the given equation: put $y=x+e$; the transformed is

$$\phi(y+e)=0 \quad \text{or} \quad \phi(e)+\phi'(e).y+\phi''(e).\frac{y^2}{1.2}+\ldots\ldots$$

$$+\phi''^{(n-1)}(e).\frac{y^{n-1}}{1.2..(n-1)}+y^n=0,$$

the dimensions of $\phi(e)\ \phi'(e).\phi''(e)\ldots.\phi''^{(n-)}(e)$ are respectively n, $(n-1)$, $(n-2)\ldots.1$, and therefore to deprive an equation of its second term, we must solve a simple equation, of its third a quadratic, &c.; but none of these transformations, except that of taking away the second term, is much used. De Gua pointed out one use of taking away the other terms, namely, if all the roots of $\phi(x)$ were real, and we make $\phi''^{(m)}(e)=0$, this equation has also real roots, for all the derived functions necessarily have them (Proposition XVII.). Let α, β, γ, &c. a, b, c, &c., α', β', γ', &c. be the roots taken in the decreasing order of magnitude of the equations

$$\phi'''^{(m-1)}(e)=0, \quad \phi'''^{(m)}(e)=0, \quad \phi'''^{(m+1)}(e)=0,$$

a lies between α, β, and makes $\phi'''^{(m-1)}(e)$ negative for any quantity greater than α would make it positive, and there is but one root α between such a quantity and a, but a also makes $\phi''^{(m+1)}(e)$ positive (Proposition XVII.). The same considerations show that b, when put for e, would make $\phi''^{(m-1)}(e)$ positive, and $\phi'''^{(m+1)}(e)$ negative, and so on; therefore, when a term is taken away from an equation with all real roots, the terms immediately preceding and subsequent must have contrary signs; conversely when a term of an equation is wanted, if the preceding and succeeding terms have the same signs, there is at least a couple of imaginary roots.

To deprive an equation of its second term, divide the coefficient of the second term by the index of the first, and put x equal to y minus this quotient.

Thus $x^6-6x^5+12x^4$, &c. $=0$; to take away the term involving x^5, put $x-1$ equal to y, or $x=y+1$.

And in the equation $x^n-ax^{n-1}+bx^{n-2}$, &c., put $x=y+\frac{a}{n}$.

Remark. The algebraical solution of the quadratic, cubic, and biquadratic, are much simplified by this transformation: let us consider the reason of this: the value of x, in the most comprehensive state in which it is capable of algebraical expression, consists of a term free from radicals, and of other terms affected by them; this single expression being required to give all the roots, can only do this by varying its radical parts according to the different values of the roots of unity; the part unaffected by any radical is therefore the same in all the roots; now the rational coefficient a being the sum of all the roots, it follows that the part in question in each root is $\frac{a}{n}$, for the radical parts in the summation must destroy each other to give a rational sum.

Hence, when we put $x-\frac{a}{n}=y$, the values of y will have one term fewer than the values of x (and consist essentially of radicals); it is for this reason, that greater simplicity attends the research of y than of x.

From this it is obvious and worth remarking, that the part of the *general expression* for the root of the equation $x^n-ax^{n-1}+bx^{n-2}-\&c.$, which is free from radicals, is always $=\frac{a}{n}$.

28. Problem 2. To transform an equation into one of which the roots are those of the given equation multiplied by a given number.

Let m be the given number $x^n-ax^{n-1}+bx^{n-2}-\&c.=0$, the given equation, put $x=\frac{y}{m}$, then by substitution, and multiplying by m^n, we have $y^n-amy^{n-1}+bm^2y^{n-2}-\&c.$

The use of this transformation refers only to the numerical solution of equations; for if any term in the equation proposed had a fractional coefficient $\frac{p}{q}$, then put $x=\frac{y}{q}$, and all the coefficients of the transformed are integers; if there were two such coefficients as $\frac{p}{q}\frac{p'}{q'}$, put $x=\frac{y}{qq'}$, and so on.

If the proposed equation had any rational root, the transformed would have an integer root; for if an equation has all its coefficients integers, it cannot have a fractional root; for suppose $\frac{r}{s}$ put for the unknown quantity, and all the terms after the first collected over a common denominator s^{n-1}, and let N be the numerator; hence, $\frac{r^n}{s^n}+\frac{N}{s^{n-1}}=0$, $\frac{r^n}{s}+N=0$, which would be impossible if $\frac{r}{s}$ were a fraction in its lowest term, since then $\frac{r^n}{s}$ would also be in its lowest terms.

Any equation which has numerically a rational root may be easily solved by this transformation.

29. Problem 3. To transform an equation into one in which the roots are the reciprocals of the roots of a given equation.

Put $x=\frac{1}{y}$, and multiply by y^n.

It will easily be seen in this transformation, that if an equation in x want the mth term from the beginning, the transformed in y will want the mth term from the end; by combining therefore this transformation with that in Problem 1, we can take away the last term but one by a simple equation, the last but two by a quadratic, &c.

If the coefficients of the terms of an equation taken from the beginning be the same as those equidistant from the end, the transformed and original equation have the same form, and therefore the same

roots; but since $y=\frac{1}{x}$, it follows that amongst the n roots of the proposed equation for each root, there is another which is its reciprocal; these equations have been called recurring or reciprocal equations.

If α, β, γ, &c., be roots of a recurring equation, then $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$, &c. also are roots; and when their number is odd, there must be at least one root which is its own reciprocal, and its value is therefore ± 1. Having separated this root if it exists, by the division of the factor $x \mp 1$, the quotient will be of a recurring form, and will be reducible to one half its dimensions by putting $x+\frac{1}{x}=y$; to facilitate the transformation we may use the formula

$$x^n+\frac{1}{x^n}=y^n-ny^{n-2}+\frac{n(n-3)}{1.2}.y^{n-4}-\frac{n(n-4)(n-5)}{1.2.3}.y^{n-6}, \text{ \&c.,}$$

which is the sum of the nth powers of the roots of the equation $x^2-yx+1=0$ by Proposition XVIII.

Example. $6x^4+35x^3+62x^2+35x+6=0$.

Hence $\quad 6\left(x^2+\frac{1}{x^2}\right)+35\left(x+\frac{1}{x}\right)+62=0$; put $x+\frac{1}{x}=y$,

$6(y^2-2)+35y+62=0$.

When $\quad y=-\frac{5}{2}$ and $-\frac{10}{3}$

Taking the first value $\quad x^2+\frac{5}{2}x+1=0$, where $x=-2$ and $-\frac{1}{2}$,

.... second .... $x^2+\frac{10}{3}x+1=0$, ... $x=-3$ and $-\frac{1}{3}$.

30. Problem 4. To transform an equation into one of which the roots shall be the mth powers of the roots of the proposed.

Let α, β, γ, &c., be the roots of the proposed $x^n-ax^{n-1}+bx^{n-2}$, &c. $=0$, then α^m, β^m, γ^m, &c. are the roots of the transformed.

Let S_1, S_2, S_3, &c. be the sums of the first, second, third, &c., powers of the roots of the proposed.

σ_1, σ_2, σ_3, &c. similar sums in the required equation; then $\sigma_1=S_m$, $\sigma_2=S_{2m}$, $\sigma_3=S_{3m}$, &c.; hence, if the required equation be $y^n-Ay^{n-1}+By^{n-2}-Cy^{n-3}$, &c. $=0$, the coefficients A, B, C, &c., may be found by Newton's formula, or by Proposition XIX.

Remarks. Since $A=S_m$, and a root of the equation in y is the mth power of that in x; and since it has been shown that the coefficient of the second term divided by the radix with sign changed is the rational part of the general formula for the root, therefore $\frac{S_m}{n}$ the rational part of the mth power of any root.

We must be careful to remember, that the root is here, as before supposed to be, in a form which comprehends all the roots, which is that given by the algebraical solution of the equation.

Thus, even though an equation be not solved algebraically, we can find the rational part of a function of its root.

31. Conversely, by comparing the rational parts of the powers of compound surds with the values of $\frac{1}{n}.S_m$ in the equation of which the surd represents the general root, we can find the quantities affected by the different surds, and thus discover the root of the equation. At present we shall not pursue this further than to give a few examples for illustration.

Example 1. Suppose $\alpha+\beta^{\frac{1}{2}}$ to be the *general* root of the equation $x^2-ax+b=0$, it is required to find α and β.

$$\left.\begin{array}{lll}\text{Rational part of } (\alpha+\beta^{\frac{1}{2}}) & = \alpha \\ \quad ,, \quad ,, \quad \text{of } (\alpha+\beta^{\frac{1}{2}})^2 & = \alpha^2+\beta\end{array}\right\}$$

Again, $S_1 = a$, $S_2 = a^2-2b$; hence, by the above principle, we have

$$\alpha = \frac{a}{2} \quad \alpha^2+\beta = \frac{a^2-2b}{2} \quad \text{or } \beta = \frac{a^2}{4}-b.$$

Example 2. We have seen that when the second term of an equation is taken away, the general root is entirely surd; suppose therefore $\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}$ to be the general root of the equation $x^3+ax-b=0$, to find α and β.

Observe that $\alpha^{\frac{1}{3}}$ being an irreducible surd, $\alpha^{\frac{2}{3}}$ is also such, for if rational and equal to γ, we should have $\alpha^{\frac{1}{3}} = \gamma^{\frac{1}{2}}$, which would give a different form for the root than that supposed.

The only part which can be rational of $(\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}})^2$ is therefore $2(\alpha\beta)^{\frac{1}{3}}$, that is, $\alpha\beta$ must be a perfect cube; and since $S_2 = -2a$, the above rational part $= \frac{S_2}{3}$ which gives $(\alpha\beta)^{\frac{1}{3}} = -\frac{a}{3}$

Again the rational part of $(\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}})^3$ is only $\alpha+\beta$ for the term $3\alpha^{\frac{2}{3}}\beta^{\frac{1}{3}} = 3(\alpha\beta)^{\frac{1}{3}}.\alpha^{\frac{1}{3}}$ and is irrational, because $(\alpha\beta)^{\frac{1}{3}}$ is rational.

But since $S_3+a\,S_1-3b=0$, therefore $S_3 = 3b$; hence $\alpha+\beta = b$ the equations for finding α, β are $\alpha\beta = -\frac{a^3}{27}$ $\alpha+\beta=b$; whence

$$\alpha = \frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)} \quad \beta = \frac{b}{2}-\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)} \text{ and the value}$$

of x is then $\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}$ subject to the condition $(\alpha\beta)^{\frac{1}{3}}\ldots$ rational.

Thus we obtain the same system of roots as that given in the *Algebraical Expressions.*

Example 3. Suppose $\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}$ to be the general root of the

biquadratic deprived of its second term $x^4+ax^2-bx+c=0$, to find α, β, γ.

$$\text{Rational part of } \left(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}\right)^2=\alpha+\beta+\gamma$$

$$\ldots\ldots \text{ of } \left(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}\right)^3=6(\alpha\beta\gamma)^{\frac{1}{2}}$$

for the simple rectangles $(\alpha\beta)^{\frac{1}{2}}$ &c. are radical, otherwise $\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}$ would be equivalent to only one square root.

$$\text{Rational part of } \left(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}\right)^4=\alpha^2+\beta^2+\gamma^2.$$

Again $S_2=-2a$, $S_3=3b$, $S_4=2a^2-4c$; therefore to find α, β, γ, we must have $\alpha+\beta+\gamma=-\dfrac{a}{2}$ $(\alpha\beta\gamma)^{\frac{1}{2}}=\dfrac{b}{8}$, $\alpha^2+\beta^2+\gamma^2=\dfrac{a^2}{2}-c.$

Hence $\alpha\beta+\alpha\gamma+\beta\gamma=\dfrac{c}{2}-\dfrac{a^2}{8}$ therefore α, β, γ are the 3 roots of the cubic $\alpha^3+\dfrac{a^2}{2}.\ \alpha^2+\left(\dfrac{c}{2}-\dfrac{a^2}{8}\right)\ \alpha-\dfrac{b^2}{64}=0$; and then $x=\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}$ subject to the condition $\alpha^{\frac{1}{2}}.\ \beta^{\frac{1}{2}}.\ \gamma^{\frac{1}{2}}..$ of same sign as b, which determines its four roots.

These are moreover the only forms of irreducible surds, by which the *general* roots of quadratic, cubic, and biquadratic equations can be respectively expressed; that which has been called the irreducible case of cubics, and which exists in reality only in the arithmetical solution can therefore never be removed by any algebraical solution seeking to express a root by irreducible surds; we have a sure test of this by the impossibility of making the rational parts of the powers of other surds to coincide with the corresponding quantities deduced from the coefficients of the equation.

It may also be remarked that the march of the index of the monomial surds as we pass from an equation of a lower to a higher degree does not follow that indicated by the dimensions of the equation, but by the prime divisions of the degree: thus the index in the quadratic is $\frac{1}{2}$, in the cubic $\frac{1}{3}$, in the biquadratic $\frac{1}{2}$, for the same method will show that it is impossible to express the general root of a biquadratic in irreducible biquadratic surds.

32. Problem 5. To transform an equation into one of which the roots are any given functions of the roots of the proposed.

Let α, β, γ, &c. be the roots of the equation $\phi(x)=0$, and $F(\alpha)$, $F(\beta)$, $F(\gamma)$, &c. the roots of the required equation; eliminate x between the equation $\phi(x)=0$, $y-F(x)=0$, by a process similar to that of finding the greatest common measure, the resulting equation in y will have the roots proposed.

Or, if we denote by $F(0)$, $F'(0)$, $F''(0)$, &c. the values which $F(x)$ and its derived functions acquire by putting $x=0$, we have

$$F(x)=F(0)+F'(0).\ x+F''(0).\ \frac{x^2}{1.2}\ \&c.$$

and therefore if S_1, S_2, &c. represent the sums of the powers of the roots of the given equation, σ_1 σ_2 &c. of the required, we have

$$\sigma_1 = n\,F(0) + F'(0).\ S_1 + \frac{F''(0).}{1.2}\ \&c.$$

$$\sigma_2 = n\{F(0)\}^2 + 2F(0).F'(0).S_1 + \Big\{\{F'(0)\}^2 + F(0).\ F''(0)\Big\}.\ S_2 + \&c.$$

&c.

And S_1, S_2..S_{12} are known by the coefficients of the given equation, while the coefficients of the terms of the transformed are known by the calculated values of σ_1, σ_2, σ^3, &c.

Remark. If the transformed equation is of the same form, that is, has the same coefficients as the original, it must have the same roots, and therefore to each root a there corresponds another $F(a)$, which is the given function of it: if the dimensions are odd, one root will be determinable by the equation $(F\,a) - a = 0$, and may be found by seeking the common measure of $\phi(x)$ and $F(x) - x$, and the other roots may be found when this is separated in the same manner as in equations of even dimensions, by putting $F(x) + x = z$, which will give an equation in z of only half the dimensions of the equation so reduced.

33. Problem 6. To find an equation of which the roots are the differences of the roots of two given equations.

Let $$x^n - ax^{n-1} + bx^{n-2} - cx^{n-3} + \&c. = 0$$
$$y^m - a'y^{m-1} + b'y^{m-2} - c'y^{m-3} + \&c. = 0$$

be the given equations, we are required to form an equation of which the roots are the excess of the roots of the second above those of the first.

Since each root of the first being subtracted from any *one* root of the second gives a root of the required, the latter must have $m.n$ roots, and its form (putting $q = m.n$) may be represented by

$$z^q - A\,z^{q-1} + B\,z^{q-2} - C\,z^{q-3} + \&c. = 0$$

Let S_1, S_2, S_3, &c. represent the sums of the first, second, third, &c. power of the roots of the equation in x.

S'_1, S'_2, S'_3, &c. similar quantities for the equation in y

and s_1, s_2, s_3, &c, the sums of the powers of the roots of the equation in z.

We have $s_1 = \Sigma z = \Sigma(y - x) = \Sigma y - \Sigma x = nS'_1 - mS_1$ for the sum Σy, is to be taken for mn terms, and the same cycle of terms recur after we have gone through the m values of y; the same remark applies to the sum of any function of y unaffected by x; but when combined as $\Sigma x^e\ y^{e'}$ this has no such cycles, and is strictly the same as $\Sigma x^e\ \Sigma y^{e'}$, under which form the pure powers may be included, for $\Sigma x^e y^0 = \Sigma x^e\ \Sigma y^0 = m\ \Sigma x^e$

And generally since

$$s_p = \Sigma\ z^p = \Sigma\,(y - x)^p$$
$$= \Sigma y^p\,x^0 - p\,\Sigma y^{p-1}\,\Sigma x + \frac{p.(p-1)}{1\ .\ 2}\,\Sigma y^{p-2}\,\Sigma x^2 - \&c.$$

therefore

$$s_p = nS'_p - pS'_{p-1}\ S_1 + \frac{p.\,(p-1)}{1\ .\ 2}S'_{p-2}\,S_2 \ldots\ldots \pm pS'_1\ S_{p-1} \pm m\,S_p$$

The quantities s_1, s_2, s_3, being calculated from this formula, the co-

efficients of the required equation are known by the theorems already given.

Remark. Suppose the two given equations to be exactly alike, the dimensions of the transformed equation would be n^2; but as it must have n roots equal to zero, it becomes of $n(n-1)$ dimensions, by dividing by x^n; moreover the roots of the transformed equation have for each an equal root with a contrary sign; therefore the sums of their odd powers vanish, the degree of the transformed equation may be then reduced to $\frac{n(n-1)}{2}$ by making $u=z^2$; the terms equidistant from both ends of the foregoing value of s_p being alike may be taken together, and therefore if U_p be the sum of the p^{th} powers of the different values of u we have

$$U_p = nS_{2p} - 2p\,S_1\,S_{2p-1} + \frac{2p(2p-1)}{1,\ 2.}\,S_2\,S_{2p-2} + \ldots\ldots$$

$$\ldots\ldots, \pm \frac{2p(2p-1)\,(2p-2)\ldots\ldots(p+1)}{1.\ \ 2.\ \ \ 3\ \ldots\ldots\ldots\ldots p}.\ \frac{(S_p)^2}{2}$$

Put $m=\frac{n(n-1)}{2}$ and form an equation

$$u^m + Au^{m-1} + Bu^{m-2} + \ldots. + Pu + Q = 0;$$

such, that the sums of the p^{th} powers of the roots may be that quantity expressed by U_p, the roots of this equation will be the squares of the differences of the roots of the equation in x; this transformation first given by Waring has yet some important uses in the solution of equations.

34. Problem 7. To find the last or absolute term in the equation whose roots are the squares of the differences of the roots of the proposed.

Let α, β, γ, &c. be the roots of the equation $\phi(x)=0$, and let $\phi'(x)$ be the derived function of $\phi(x)$, then since

$$\phi'(x) = (x-\beta)\,(x-\gamma)\ \ldots + (x-\alpha)\,(x-\gamma)\ldots.$$
$$+ (x-\alpha)\,(x-\beta)\,(x-\delta)\ldots.\&c.$$

therefore $\phi'(\alpha) = (\alpha-\beta)\,(\alpha-\gamma)\,(\alpha-\delta)\ldots\ldots$

and similarly $\phi'(\beta) = (\beta-\alpha)\,(\beta-\gamma)\,(\beta-\delta)$

$\phi'(\gamma) = (\gamma-\alpha)\,(\gamma-\beta)\,(\gamma-\delta)$

&c. &c.

Hence

$$\phi'(\alpha).(\phi'(\beta).\phi'(\gamma)\ldots. = (-1)^{\frac{n(n-1)}{2}}.(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2\ldots..$$

This is the required last term.

The product $\phi'(\alpha)\,.\,\phi'(\beta)\,.\,\phi'(\gamma)\ldots\ldots$ being a symmetrical function of the roots may be readily expressed by means of the coefficients of the proposed equation, and also by means of those of the derived function $\phi'(x)$ by the following consideration.

Let α_1, β_1, γ_1, &c. be the roots of the equation $\phi'(x)=0$, then since $\phi'(x) = n\,(x-\alpha_1)\,(x-\beta_1)\,(x-\gamma_1)\ldots.\&c.$

therefore $\phi'(\alpha)\,.\,\phi'(\beta)\,.\,\phi'(\gamma)\ldots = n^n\,(\alpha-\alpha_1)\,(\beta-\alpha_1)\,(\gamma-\alpha_1)\ldots$
$\times(\alpha-\beta_1)\,(\beta-\beta_1)\,(\gamma-\beta_1)\ldots\times(\alpha-\gamma_1)\,(\beta-\gamma_1)\,(\gamma-\gamma_1)\ldots.\times\&c.$

$$\text{But}\qquad (\alpha-\alpha_1)(\beta-\alpha_1)(\gamma-\alpha_1)\ldots\ldots=(-1)^n\phi(\alpha_1)$$
$$(\alpha-\beta_1)(\beta-\beta_1)(\gamma-\beta_1)\ldots\ldots=(-1)^n\phi(\beta_1)$$
$$\&c. \;\&c.$$

and observing that $(-1)^{n(n-1)}=1$, we obtain

$$\phi'(\alpha)\,.\,\phi'(\beta)\,.\,\phi'(\gamma)\ldots\ldots=n^n\phi(\alpha_1)\,.\,\phi(\beta_1)\,.\,\phi(\gamma_1)\ldots\ldots$$

the last product being a symmetrical function of the roots of the desired equation.

35. In order that an equation may have two equal roots, a certain relation of the coefficients is necessary; this relation, which expresses their mutual dependence, must be such that the last term in the equation *to the squares of the differences* above obtained may be zero, since one of the roots of the latter equation is nothing; the condition therefore that the equation $\phi(x)=0$ may have equal roots, is $\phi'(\alpha)$. $\phi'(\beta)\,.\,\phi'(\gamma)\ldots\ldots=0$, or which is the same $\phi(\alpha_1)\,.\,\phi(\beta_1)\,.\,\phi(\gamma_1)\ldots=0$; the latter is preferable in practice for having fewer factors.

Example 1. Required the condition that x^2+ax+b may have two equal roots.

$$\phi(x)=x^2+ax+b.\quad \phi'(x)=2x+a$$

Hence $\qquad \alpha_1=-\dfrac{a}{2}$

therefore $\quad \phi(\alpha_1)=\left(-\dfrac{a}{2}\right)^2-\dfrac{a^2}{2}+b=-\dfrac{a^2}{4}+b$

the required condition in this case is $b-\dfrac{a^2}{4}=0$

Example 2. To find the condition that the cubic $x^3+ax^2+bx+c=0$ may have two equal roots.

$$\text{Here}\quad \phi(x)=x_3+ax^2+bx+c\qquad \phi'(x)=3x^2+2ax+b$$

$$\text{or}\; x^2+\frac{2a}{3}\,.\,x+\frac{b}{3}=(x-\alpha_1)(x-\beta_1)$$

we must have $(\alpha_1^3+a\alpha_1^2+b\alpha_1+c)(\beta_1^3+a\beta_1^2+b\beta_1+c=0$

hence $\alpha_1^3\beta_1^3+a\alpha_1^2\beta_1^2(\alpha_1+\beta_1)+b\alpha_1\beta_1(\alpha_1^2+\beta_1^2)+c(\alpha_1^3+\beta_1^3)$
$+a^2\alpha_1^2\beta_1^2+ab\alpha_1\beta_1(\alpha_1+\beta_1)+ac(\alpha_1^2+\beta_1^2)+b^2\alpha_1\beta_1+bc(\alpha_1+\beta_1)$
$+c^2=0$

Put $\dfrac{b}{3}$ for $\alpha_1\beta_1$ and collect the terms which multiply the same sums of the powers of $\alpha_1\,\beta_1$; this gives

$$\left(\frac{10b^3}{27}+\frac{a^2b^2}{9}+c^2\right)+\left(\frac{4ab^2}{9}+bc\right)(\alpha_1+\beta_1)+\left(\frac{b^2}{3}+ac\right)(\alpha_1^2+\beta_1^2)$$
$$+c(\alpha_1^3+\beta_1^3)=0$$

where $\alpha_1+\beta_1=-\dfrac{2a}{3}\quad \alpha_1^2+\beta_1^2=\dfrac{4a^2}{9}-\dfrac{2b}{3}\quad \alpha_1^3+\beta_1^3=-\dfrac{8a^3}{27}+\dfrac{2ab}{3}$

by substituting these values the required condition becomes

$$c^2+\frac{2ac}{27}(2a^2-9b)+\frac{b^2}{27}(4b-a^2)=0.$$

When $c=0$, the proposed equation becomes $(x^2+ax+b).x=0$ and the condition for the equality of two of its roots is $b^2(4b-a^2)=0$, which is satisfied either by supposing $b=0$, or $a^2-4b=0$; on the first supposition the two equal roots are zero, and the second condition is that already found for the equality of the roots of the quadratic equation $x^2+ax+b=0$.

It is usual to suppose $a=0$, which simplifies the proposed cubic, and the condition for equal roots is then $c^2+\dfrac{4b^3}{27}=0$.

36. Problem 8. To eliminate x between the equations $\phi(x)+y=0$ $F(x)=0$ these functions being rational and integral.

Let α, β, γ, &c. be the roots of the first, and α_1, β_1, γ_1, &c. of the second equation, then the condition $\{y+\phi(\alpha_1)\}.\{y+\phi(\beta_1)\}.\{y+\phi(\gamma_1)\}\ldots=0$, implies that $y+\phi(x)=0$ is true simultaneously with the equation $F(x)=0$: it remains to reduce this product to the form $y^m+ay^{m-1}+by^{m-2}+$ &c.$=0$, where m, which represents the number of factors, is manifestly the same as the dimensions of $F(x)$.

Taking the logarithms of these identical expressions, after dividing each by y^m, we obtain

$$\text{Log.}\left\{1+\frac{a}{y}+\frac{b}{y^2}+\ldots+\frac{p}{y^{m-1}}+\frac{q}{y^m}\right\}$$

$$=\text{Log.}\left\{1+\frac{\phi(\alpha_1)}{y}\right\}+\text{Log.}\left\{1+\frac{\phi(\beta_1)}{y}\right\}+\text{Log.}\left\{1+\frac{\phi(\gamma_1)}{y}\right\}+$$

$$=\Sigma\,\text{Log.}\left\{1+\frac{(\phi\alpha_1)}{y}\right\}\text{ for abridgment}$$

$$=\Sigma\left\{\frac{\phi\alpha_1}{y}-\tfrac{1}{2}\frac{(\phi\alpha_1)^2}{y^2}+\tfrac{1}{3}\frac{(\phi\alpha_1)^3}{y^3}-\text{\&c.}\right\}$$

But since α_1 is a root of the equation $F(x)=0$, we have

$$\Sigma\phi(\alpha_1)=\text{coefficient of }\tfrac{1}{x}\text{ in }-\phi'(x)\ \text{Log.}\frac{F(x)}{x^m}$$

$$\Sigma\{\phi(\alpha_1)\}^2=\ldots\ldots\ldots\text{ in }-2\phi(x)\,\phi'(x)\ \text{Log.}\frac{F(x)}{x^m}$$

$$\Sigma\{\phi(\alpha_1)\}^3=\ldots\ldots\ldots\text{ in }-3\phi(x)^2\phi'(x)\ \text{Log.}\frac{F(x)}{x^m}$$

&c.

and also observing that

$$-\frac{\phi'x}{y}-\frac{\phi(x).\phi'(x)}{y^2}-\frac{(\phi x)^2\,\phi'(x)}{y^3}-\text{\&c.}=\frac{-\phi'(x)}{y+\phi(x)}$$

therefore $\text{Log.}\left\{1+\dfrac{a}{y}+\dfrac{b}{y^2}+\ldots\dfrac{+p}{y^{m-1}}+\dfrac{q}{y^m}\right\}=$ coefficient of $\frac{1}{x}$ in

$$\frac{-\phi'(x)}{y+\phi(x)}\ \text{Log.}\ \frac{F(x)}{x^m}$$

Well $y+\phi(x)=(x-\alpha)(x-\beta)(x-\gamma)\ldots\ldots$taking the coefficient of the highest power of x to be unity.

Hence $$\frac{-\varphi'(x)}{y+\varphi(x)}=\frac{1}{\alpha-x}+\frac{1}{\beta-x}+\frac{1}{\gamma-x}+\&c.$$

$$=\quad\frac{1}{a-x}\text{ for abridgment}$$

$$=\Sigma\, a^{-1}+x\,\Sigma\, a^{-2}+x^2\,\Sigma\, a^{-3}+\&c.$$

Also

$$\text{Log.}\,\frac{\mathrm{F}(x)}{x^m}=\text{Log.}\left(1-\frac{\alpha_1}{x}\right)+\text{Log.}\left(1-\frac{\beta_1}{x}\right)+\text{Log.}\left(1+\frac{\gamma_1}{x}\right)+\&c.$$

$$=\Sigma\,\text{Log.}\left(1-\frac{\alpha_1}{x}\right)$$

$$=-\left\{\frac{1}{x}\,\Sigma\,\alpha_1+\frac{1}{2}\cdot\frac{1}{x^2}\,\Sigma\,\alpha_1^2+\frac{1}{3}\cdot\frac{1}{x^3}\,\Sigma\,\alpha_1^3+\&c.\right\}$$

and if the corresponding terms in both series are multiplied, the products will each contain $\frac{1}{x}$; wherefore

$$\text{Log.}\left\{1+\frac{a}{y}+\frac{b}{y^2}+\ldots+\frac{p}{y^{m-1}}+\frac{q}{y^m}\right\}$$

$$=-\left\{\Sigma\,\alpha_1\,\Sigma\,a^{-1}+\frac{1}{2}\,\Sigma\,\alpha_1^2\,\Sigma\,a^{-2}+\frac{1}{3}\cdot\frac{1}{x^3}\,\Sigma\,a_1^3+\&c.\right\}$$

where y being of n dimensions, we may stop in seeking a at the term $\frac{1}{n}\,\Sigma\,\alpha_1^n\,\Sigma\,a^{-n}$ and for b at the term $\frac{1}{2n}\,\Sigma\,\alpha_1^{2n}\,\Sigma\,\alpha^{-2n}$, and so on.

The exact expression for $y^m+ay^{m-1}+by^{m-2}$ &c. is $y^m\cdot\varepsilon^{-\Sigma'\frac{1}{n}\,\Sigma\alpha_1^n\,\Sigma\alpha^{-n}}$; the sign Σ' denoting the sum from $n=1$ to $n=\infty$

Corollary. Let the form of the function $\mathrm{F}(x)$ be x^m-1; then if m' be not a multiple of m, we have $\Sigma\alpha^{m'}=0$, and when it is a multiple, $\Sigma\alpha^{m'}=m$; hence

Log.

$$\left\{1+\frac{a}{y}+\frac{b}{y^2}+\ldots+\frac{p}{y^{m-1}}+\frac{q}{y^m}\right\}=\frac{\Sigma\phi(\alpha_1)}{y}-\frac{1}{2}\cdot\frac{\Sigma(\phi\alpha_1)^2}{y^2}+\frac{1}{3}\,\frac{\Sigma(\phi\alpha_1)^3}{y^3}-\&c.$$

$$=-\Sigma\{\alpha^{-m}+\alpha^{-2m}+\alpha^{-3m}+\&c.\}$$

if the former series be used, we may omit in $(\phi\alpha_1)^{m'}$ those terms in which the index is not a multiple of m, and put m for any term in which the index of α_1 is a multiple of m, and equate like powers of y; if we use the latter series we may equate like powers of y, taking always one term more under the sign of Log. than under that of Σ: thus,

Coefficient of

$$\frac{1}{y}\text{ in Log.}\left\{1+\frac{a}{y}\right\}\quad=\text{coefficient of }\frac{1}{y}\text{ in }\Sigma a^{-m},$$

Coefficient of

$$\frac{1}{y^2} \text{ in Log.}\left\{1 + \frac{a}{y} + \frac{b}{y^2}\right. = \text{coefficient of } \frac{1}{y^2} \text{ in } \Sigma(\alpha^{-m} + \alpha_1^{-2m}),$$

&c. &c.

and the exact expression for $y^m + ay^{m-1} + by^{m-2}$, &c., is $\varepsilon^{-\Sigma\left(\frac{1}{\alpha^m - 1}\right)}$ where ε stands for the base of Naperian logarithms.

When $\phi(x)$ is of lower dimensions than m, we have $a=0$; for then $\Sigma\phi(\alpha_1)$ evidently vanishes.

Generally, let k_r be the sum of the coefficients of x^m, x^{2m}, &c., in $(\phi x)^r$, then,

$$1 + \frac{a}{y} + \frac{b}{y^2} + \text{\&c.} = \left\{1 + \frac{mk_1}{y} + \frac{m^2k_1^2}{1.2.y^2} + \frac{m^3k_1^3}{1.2.3.y^3}, \text{\&c.}\right\}$$

$$+ \left\{1 + \frac{mk_2}{y^2} + \frac{m^2k_2^2}{1.2.y^4} + \text{\&c.}\right\} + \text{\&c.}$$

thus

$$a = mk_1,\ b = \frac{m^2k_1^2}{1.2} + mk_2,\ c = \frac{m^3k_1^3}{1.2.3} + m^2k_1k_2 + mk_3, \text{ \&c.}$$

37. We shall now proceed to the solution of equations in finite surds, first by simple individual methods, and secondly on general principles, adding remarks on each.

CUBIC EQUATIONS.

The general form of a cubic equation is $x^3 + ax^2 + bx + c = 0$, a, b, c, being known quantities, positive or negative.

The simplest equation in this class is $x^3 - 1 = 0$, one root of which being $x=1$, we may reduce $x^3 - 1$ to the form $(x-1)(x^2+x+1)$; the other two roots are given by the equation $x^2 + x + 1 = 0$, and are easily found to be $x = \frac{-1+\sqrt{-3}}{2}$, $x = \frac{-1-\sqrt{-3}}{2}$, each of which is the square of the other; for if we call one of them α, then since $\alpha^3 = 1$, we have by squaring $(\alpha^2)^3 = 1$; therefore, α^2 is a root as well as α.

If there is proposed $x^3 = b$, and α denote, as before, one of the imaginary cube roots of unity, and $\sqrt[3]{b}$ the arithmetical cube root of b, the three values of x are then $\sqrt[3]{b}$, $\alpha\sqrt[3]{b}$ $\alpha^2\sqrt[3]{b}$, for the cubes of any of these quantities is b.

Thus, let $x^3 = -1$, then $x = -1$, is the real root, and the two imaginary roots are $-\alpha - \alpha^2$.

Given $x^3 + ax + b = 0$.

Put x, which is no longer a simple cube root, equal to the sum of two cube roots; that is, let $x = p^{\frac{1}{3}} + q^{\frac{1}{3}}$.

Then, $$x^3 = p + 3p^{\frac{2}{3}}q^{\frac{1}{3}} + 3p^{\frac{1}{3}}q^{\frac{2}{3}} + q,$$

$$= (p+q)+3(pq)^{\frac{1}{3}}\left(p^{\frac{1}{3}}+q^{\frac{1}{3}}\right),$$

and $$ax+b= \quad b+ \quad a\left(p^{\frac{1}{3}}+q^{\frac{1}{3}}\right).$$

We thus have $(p+q+b)+\left\{3(pq)^{\frac{1}{3}}+a\right\}\left(p^{\frac{1}{3}}+q^{\frac{1}{3}}\right)=0,$

which equation, containing two indeterminate quantities, allows us to make another condition between p and q; namely,

$$(pq)^{\frac{1}{3}} = -\frac{a}{3}.$$

Hence p and q may be determined by the equations

$$p+q=-b, \qquad pq=-\frac{a^3}{27};$$

or
$$\begin{cases} p^2+2pq+q^2= \quad b^2 \\ 4pq \quad =-\dfrac{4a^3}{27} \end{cases}$$

therefore $p^2-2pq+q^2=4\left(\dfrac{b^2}{4}+\dfrac{a^3}{27}\right),$

$$p \mathbin{\dot{-}} q = 2\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)};$$

whence $p=\dfrac{b}{2}+\sqrt{\left(\dfrac{b^2}{4}+\dfrac{a^3}{27}\right)}; \quad q=\dfrac{b}{2}-\sqrt{\left(\dfrac{b^2}{4}+\dfrac{a^3}{27}\right)},$

and $x=p^{\frac{1}{3}}+q^{\frac{1}{3}}.$

Now, $p^{\frac{1}{3}}$ admits also of the values $\alpha p^{\frac{1}{3}}$, $\alpha^2 p^{\frac{1}{3}}$,

$q^{\frac{1}{3}}$ .... of .... $\alpha^2 q^{\frac{1}{3}}$, $\alpha q^{\frac{1}{3}}$,

and the products of the pairs in the same vertical lines arise each the same as $p^{\frac{1}{3}}.q^{\frac{1}{3}}$, since $\alpha^3=1$; that is, they satisfy the condition imposed in the process of investigation; viz. $(pq)^{\frac{1}{3}}=-\dfrac{a}{3}$ a rational quantity, admitting of only one value.

Hence the 3 roots of the cubic equation $x^3+ax+b=0$ are

$$x=p^{\frac{1}{3}}+q^{\frac{1}{3}}, \quad x=\alpha p^{\frac{1}{3}}+\alpha^2 q^{\frac{1}{3}}, \quad x=\alpha^2 p^{\frac{1}{3}}+\alpha q^{\frac{1}{3}},$$

where α is an imaginary cube root of unity, and p, q have the values above assigned.

If the proposed equation has equal roots, the condition for their existence is
$$\frac{b^2}{4}+\frac{a^3}{27}=0;$$

we have then $p=q=\dfrac{b}{2}=\sqrt{-\dfrac{a^3}{27}};$

therefore, $p^{\frac{1}{3}}=q^{\frac{1}{3}}=\sqrt{-\dfrac{a}{3}};$

and the values of x are

$$x = p^{\frac{1}{3}} + q^{\frac{1}{3}} = 2\sqrt{-\frac{a}{3}},$$

$$x = \alpha p^{\frac{1}{3}} + \alpha^2 q^{\frac{1}{3}} = (\alpha+\alpha^2)\sqrt{-\frac{a}{3}} = -\sqrt{-\frac{a}{3}},$$

$$x = \alpha^2 p^{\frac{1}{3}} + \alpha q^{\frac{1}{3}} = (\alpha^2+\alpha)\sqrt{-\frac{a}{3}} = -\sqrt{-\frac{a}{3}},$$

the latter two being the equal pair.

38. It will be useful, in connexion with the general theory of equations, to add some remarks on the roots thus obtained for the equation x^3+ax+b, for which purpose let

$$x_1 = p^{\frac{1}{3}} + q^{\frac{1}{3}}, \qquad x_2 = \alpha p^{\frac{1}{3}} + \alpha^2 q^{\frac{1}{3}}, \qquad x_3 = \alpha^2 p^{\frac{1}{3}} + \alpha q^{\frac{1}{3}};$$

hence, $x_1+x_2+x_3 = (1+\alpha+\alpha^2)\left(p^{\frac{1}{3}}+q^{\frac{1}{3}}\right)$, and by the known properties of the roots of unity we have $1+\alpha+\alpha^2=0$; therefore,

$$x_1+x_2+x_3=0.$$

Again, $x_1 x_2 = \alpha p^{\frac{2}{3}} + (\alpha+\alpha^2)\, p^{\frac{1}{3}} q^{\frac{1}{3}} + \alpha^2 q^{\frac{2}{3}},$

$$x_1 x_3 = \alpha^2 p^{\frac{2}{3}} + (\alpha+\alpha^2)\, p^{\frac{1}{3}} q^{\frac{1}{3}} + \alpha q^{\frac{2}{3}},$$

$$x_2 x_3 = p^{\frac{2}{3}} + (\alpha+\alpha^2)\, p^{\frac{1}{3}} q^{\frac{1}{3}} + q^{\frac{2}{3}}; \text{ since } \alpha^3=1, \text{ and } \alpha^4=\alpha.$$

Hence, $x_1x_2+x_1x_3+x_2x_3 = 3(\alpha+\alpha^2)\, p^{\frac{1}{3}} q^{\frac{1}{3}} = -3(pq)^{\frac{1}{3}},$

which agrees with the supposition we had made, that $(pq)^{\frac{1}{3}} = -\frac{a}{3}$.

Thirdly, $x_1x_2x_3 = p + (1+\alpha+\alpha^2)\left(p^{\frac{2}{3}} q^{\frac{1}{3}} + p^{\frac{1}{3}} q^{\frac{2}{3}}\right) + q = p+q.$

Again, from the equations which express x_1 and x_2 we have

$$\alpha^2 x_1 - x_2 = (\alpha^2-\alpha)\, p^{\frac{1}{3}}, \qquad \alpha x_1 - x_2 = (\alpha-\alpha^2)\, q^{\frac{1}{3}};$$

similarly $\quad \alpha x_1 - x_3 = (\alpha-\alpha^2)\, p^{\frac{1}{3}}, \qquad \alpha^2 x_1 - x_3 = (\alpha^2-\alpha)\, q^{\frac{1}{3}},$

$$x_2 - \alpha x_3 = (\alpha-1)\, p^{\frac{1}{3}}, \qquad \alpha x_2 - x_3 = (1-\alpha) q^{\frac{1}{3}}.$$

Put $-x_1-x_2$ instead of x_3 in the second and third systems of equations, and observing that $\alpha+1=-\alpha^2$, $\alpha^2+1=-\alpha$, the second system is the same as the first, and the third gives

$$(\alpha+1)x_2 + \alpha x_1 = (\alpha-1)p^{\frac{1}{3}}; \qquad (\alpha+1)x_2 + x_1 = (1-\alpha)\, q^{\frac{1}{3}},$$

which, being multiplied by α, reproduces also the first system.

But since α is one of the imaginary cube roots of unity, we shall obtain a second system of values of $p^{\frac{1}{3}}$, $q^{\frac{1}{3}}$, by writing α^2, the other imaginary cube root, for α in the first system; viz.,

$$\alpha^2 x_1 - x_2 = (\alpha^2-\alpha)p^{\frac{1}{3}}, \quad \alpha x_1 - x_2 = (\alpha-\alpha^2)\, q^{\frac{1}{3}},$$

which gives $\quad ax_1-x_2=(a-a^2)p^{\frac{1}{3}},\quad a^2x_1-x_2=(a^2-a)q^{\frac{1}{3}}$:

thus $p^{\frac{1}{3}}$ and $q^{\frac{1}{3}}$ merely interchange values.

By cubing we have

$$x_1^3-3ax_1^2x_2+3a^2x_1x_2^2-x_2^3=-3(a^2-a)p,$$
$$x_1^3-3a^2x_1^2x_2+3ax_1x_2^2-x_2^3=\quad 3(a^2-a)q\,;$$

and if we suppose $x_1=x_2$ we should have $p=q$, which would also result from the suppositions $x_1=x_3$, $x_2=x_3$; thus, *the condition for equal roots in the proposed cubic descends to the reducing quadratic.* Now, the quantity under the sign of square root in the solution of the quadratic, gives the condition for equal roots in that quadratic; therefore the quantity under the same sign in the solution of a cubic. by being equated with zero, gives the condition for two equal roots; in fact, we have found this condition to be $\frac{a^2}{4}+\frac{a^3}{27}=0$.

We may, *à posteriori*, infer from these considerations the form of the surds which enter the root, the condition for equal roots which results by eliminating x between $x^3+ax+b=0$, and $3x^2+a=0$ is of two dimensions in b, or of 6 in reference to the roots; the sign of $\sqrt{\ }$, which comes over the form of this condition $\frac{b^2}{4}+\frac{a^3}{27}$ reduces it to 3 dimensions; a cubic root on this square root, added to or subtracted from another term of 3 dimensions, would reduce all to similarity with the roots.

With respect to the arithmetical application of the solution of a cubic deduced from the preceding algebraical solution, it is easily seen that it will be impracticable by the mere extraction of roots when all are real (except two are equal), and practicable when two roots are imaginary; for putting for a a^2 their values $\frac{-1+\sqrt{-3}}{2}$ $\quad\frac{-1-\sqrt{-3}}{2}$, we find p and q imaginary when x_1 and x_2 are real; but if x_1 x_2 be of the form $m+n\sqrt{-1}$, $m-n\sqrt{-1}$, then the above expressions for p and q are easily found to be real.

39. Given $\quad x^3+ax^2+bx+c=0,\quad$ to find x

Put $\quad x+\frac{a}{3}=z$, from whence we have

$$\begin{aligned}
x^3&=z^3-az^2+\frac{a^2}{3}z-\frac{a^3}{27},\\
ax^2&=\quad az^2-\frac{2a^2}{3}z+\frac{3a^3}{27},\\
bx&=\qquad\qquad bz-\frac{ab}{3},\\
c&=\qquad\qquad\qquad c.
\end{aligned}$$

Let $a'=b-\frac{a^2}{3}\qquad b'=c-\frac{ab}{3}+\frac{2a^3}{27}$;

hence $z^3+a'z+b'=0$,

therefore $z=p^{\frac{1}{3}}+q^{\frac{1}{3}}$,

where
$$p=\frac{b'}{2}+\sqrt{\left(\frac{b'^2}{4}+\frac{a'^3}{27}\right)},$$
$$q=\frac{b'}{2}-\sqrt{\left(\frac{b'^2}{4}+\frac{a'^3}{27}\right)}.$$

Now since $\frac{b'}{2}=\frac{c}{2}-\left(\frac{ab}{6}-\frac{a^3}{27}\right)$,

therefore $\frac{b'^2}{4}=\frac{c^2}{4}-\left(\frac{ab}{6}-\frac{a^3}{27}\right)c+\frac{a^2}{9}\left(\frac{b^2}{4}-\frac{a^2b}{9}+\frac{a^4}{81}\right)$;

and since $\frac{a'}{3}=\frac{b}{3}-\frac{a^2}{9}$,

therefore $\frac{a'^3}{27}=\frac{b^3}{27}-\frac{a^2}{9}\left(\frac{b^2}{3}-\frac{a^2b}{9}+\frac{a^4}{81}\right)$;

whence $\frac{b'^2}{4}+\frac{a'^3}{27}=\frac{c^2}{4}-\left(\frac{ab}{6}-\frac{a^3}{27}\right)c+\frac{b^2}{27}\left(\frac{b}{3}-\frac{a^2}{4}\right)$;

therefore
$$x=-\frac{a}{3}+\left\{\frac{c}{2}-\frac{a}{3}\left(\frac{b}{2}-\frac{a^2}{9}\right)+\left\{\frac{c^2}{4}-\frac{ac}{3}\left(\frac{b}{2}-\frac{a^2}{9}\right)+\frac{b^2}{27}\left(b-\frac{a^2}{4}\right)\right\}^{\frac{1}{2}}\right\}^{\frac{1}{3}}$$
$$+\left\{\frac{c}{2}-\frac{a}{3}\left(\frac{b}{2}-\frac{a^2}{9}\right)-\left\{\frac{c^2}{4}-\frac{ac}{3}\left(\frac{b}{2}-\frac{a^2}{9}\right)+\frac{b^2}{27}\left(b-\frac{a^2}{4}\right)\right\}^{\frac{1}{2}}\right\}^{\frac{1}{3}}.$$

This apparently complicated value of x may be reduced to one of great simplicity, never before noticed, to my knowledge, by the following considerations:

If the given equation had two equal roots, the cubic in z would also have two equal roots, the reducing quadratic would have two equal roots, the term under the sign $\sqrt{}$ in the solution of that quadratic would vanish; therefore, the equating with zero the term under the sign $\sqrt{}$ in the solution of a cubic gives the condition for equal roots as well as does the equation resulting by eliminating x between the proposed and its first derived equation, their left hand members being of the same dimensions, can only differ by a numerical factor; in fact, let
$$\phi=x^3+ax^2+bx+c=0,$$
$$\phi'=3x^2+2ax+b\quad=0,$$
the result of the elimination of x arranged according to the powers of c, with unity for the coefficient of the highest power of c has been found to be
$$c^2+\frac{2ac}{27}(2a^2-9b)+\frac{b^2}{27}(4b-a^2)=0;$$
this resulting by the elimination of x, between $\phi=0$ and $\phi'=0$, may be represented in reference to this origin, by
$$[\phi,\ \phi']=0.$$

Now, comparing the quantity $[\phi, \phi']$ with that under the sign $\sqrt{}$ in the above solution of the general cubic, we find the latter to be exactly $\frac{1}{4}[\phi, \phi']$, differing only from the former in its numerical factor.

Next suppose the given equation has 3 equal roots; then its left member is a perfect cube, and we get $x=-\frac{a}{3}$; the total quantity under the sign $\{\ldots.\}^{\frac{1}{3}}$ in the general solution must therefore vanish, but the part of this under the sign $\sqrt{}$ already vanishes from the condition of two equal roots; therefore, the part outside $\sqrt{}$ being equated to zero, gives the additional condition necessary for a 3rd equal root, and being of the same dimensions, must be equivalent to that obtained by eliminating x between the proposed and its second derived, differing only by a numerical factor; in fact,

$$\phi = x^3+ax^2+bx+c=0,$$

gives
$$\phi''=6r+2a;$$

whence $[\phi, \phi''] = c - \frac{ab}{3} + \frac{2a^3}{27}$, making always the coefficient of the highest power of c unity: now, by comparing this with the part outside the sign $\sqrt{}$ in the cubic radicals, we find that part exactly equal to $\frac{1}{2}[\phi, \phi'']$, thus differing only by a numerical factor; the actual solution in this form is remarkably expressive.

If $\phi=x^3+ax^2+bx+c=0$,

then $x = -\frac{a}{3} + \left\{\frac{1}{2}[\phi,\phi''] + \frac{1}{2}\sqrt{[\phi, \phi'']}\right\}^{\frac{1}{3}} + \left\{\frac{1}{2}[\phi, \phi''] - \frac{1}{2}\sqrt{[\phi, \phi']}\right\}^{\frac{1}{3}}$.

The quantities $[\phi, \phi']$, $[\phi, \phi'']$ are readily expressed in symbolical functions of the roots; let α, β, γ be the roots of $\phi(x)=0$, then the condition for the coexistence of the equations $\phi(x)=0$, $\phi'(x)=0$ is $\phi'(\alpha).\phi'(\beta).\phi'(\gamma)=0$; and since the coefficient of c^2, that is $(\alpha\beta\gamma)^2$ is unity in $[\phi, \phi]$, we have

$$[\phi, \phi'] = \frac{1}{27}\phi'(\alpha).\phi'(\beta).\phi'(\gamma).$$

similarly $$[\phi, \phi''] = \frac{\phi''(\alpha)}{6}.\frac{\phi''(\beta)}{6}.\frac{\phi''(\gamma)}{6}.$$

Now $\phi'(\alpha)=(\alpha-\beta)(\alpha-\gamma)$, $\phi'(\beta)=(\beta-\alpha)(\beta-\gamma)$;
$$\phi'(\gamma)=(\gamma-\alpha)(\gamma-\beta);$$

therefore $$[\phi, \phi'] = -\frac{1}{27}+\{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)\}^2,$$

from whence we easily see the cause of the arithmetical failure when α, β, γ are real and unequal, and of its success when two of them are imaginary.

Moreover, $$\frac{\phi''(x)}{6}=x+\frac{a}{3}=x-\frac{\alpha+\beta+\gamma}{3};$$

therefore $[\phi, \phi'']=\frac{1}{27}(2\alpha-\beta-\gamma)(2\beta-\alpha-\gamma)(2\gamma-\alpha-\beta)$; hence α, β, and γ are simultaneously expressed by a symmetrical function of the roots.

In like manner, for a quadratic if $\phi=x^2+ax+b$, $\phi'=2x+a$, and x be eliminated so that the coefficient of the highest power of b be unity, we have $[\phi, \phi']=b-\frac{a^2}{4}$, $x=-\frac{a}{2}+\sqrt{-[\phi, \phi']}$, where $[\phi, \phi']=\frac{\phi'(\alpha)}{2}\cdot\frac{\phi'\beta}{2}$ $=-(\beta-\alpha)^2$.

40. In making arithmetical usage of the formula

$$x=\left\{-\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}^{\frac{1}{3}}+\left\{-\frac{b}{2}-\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}^{\frac{1}{3}}$$

in the equation x^3+ax+b, it will be sufficient to compute one of the cubic surds, the other will be given by the equation

$$p^{\frac{1}{3}}.q^{\frac{1}{3}}=-\frac{a}{3},$$

and the imaginary roots will be $p\alpha+q\alpha^2$, $p\alpha^2+q\alpha$, or putting for α its value, they become $-\frac{p+q}{2}\pm\frac{p-q}{2}.\sqrt{-3}$; from whence it follows, that when $p^{\frac{1}{3}}$ is rational (the coefficient a being supposed also rational) $q^{\frac{1}{3}}$ is rational, and therefore the part of the imaginary roots will be a rational quantity multiplied by $\sqrt{-3}$. Now, since a cubic equation may be formed in which the surd part of the imaginary roots is any whatever, if we suppose it irreducible to $\sqrt{-3}$, it follows that $p^{\frac{1}{3}}$ cannot be rational, nor, consequently, $q^{\frac{1}{3}}$, and yet their sum may be rational, for its value is double the real part of the imaginary roots with a contrary sign; therefore, though the sum of two irreducible quadratic surds cannot be rational the sum of two cubic surds may.

We give an example.

Let $1+\sqrt{-1}$, $1-\sqrt{-1}$, and -2 be the roots;

$$\text{then } a=-2,\ b=+4,\ \frac{b^2}{4}+\frac{a^3}{27}=4-\frac{8}{27}=\frac{100}{27}\ \therefore$$

$$x=\sqrt[3]{-2+\frac{10}{3\sqrt{3}}}-\sqrt[3]{\left(2+\frac{10}{3\sqrt{3}}\right)}=-2;$$

but since it has been proved that

$$[\phi, \phi']=-\frac{1}{27}\{(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2\},$$

it is visible that when the parts real and imaginary of α, β, γ are commensurate, or do not involve $\sqrt{-3}$, then $\sqrt{3}$ will enter in the summable cubic surds.

BIQUADRATIC EQUATIONS.

41. The method of Thomas Simpson, an analyst of first-rate genius, affords perhaps the most direct and easy solution of the biquadratic ever given; with slight variation it is as follows—

Given $x^4+ax^3+bx^2+cx+d=0$, to find x.

Add to each side a quadratic function Bx^2+Cx+D, and let B, C, D be so determined that both sides may be exact squares; then, extracting the square root, we shall have two quadratic equations to determine the four values of x.

Thus, $x^4+ax^3+(b+B)x^2+(c+C)x+(d+D)=Bx^2+Cx+D$.

The second side will be a complete square of $4BD=C^2$.... (1).

The first side may be compared with $\left(x^2+\frac{a}{2}x+m\right)^2$, or

$$x^4+ax^3+\left(2m+\frac{a^2}{4}\right)x^2+amx+m^2;$$

whence $\quad B=2m+\left(\frac{a^2}{4}-b\right),\quad C=am-c,\quad D=m^2-d.$

The equation (1) serves to determine m, as it gives

$$\left\{m+\frac{1}{2}\left(\frac{a^2}{4}-b\right)\right\}.(m^2-d)=\frac{1}{8}(am-c)^2.$$

$$\text{Or,}\quad m^3-\frac{b}{2}m^2+\left(\frac{ac}{4}-d\right)m-\left\{\frac{c^2}{8}+\frac{d}{2}\left(\frac{a^2}{4}-b\right)\right\}=0.$$

m being known from this cubic, B, C, D are known by the equations above given, and extracting the square root we have

$$x^2+\frac{a}{2}x+m=\pm(\sqrt{B}.x+\sqrt{D}),$$

which furnishes two quadratic equations for finding x.

42. We shall now give Euler's method, previously showing the reason which justifies the assumption for the form of the roots.

Let $x^4+ax^3+bx^2+cd+d=0$.

The rational part of the root, as has been shown before, must be $-\frac{a}{4}$; put therefore $x=-\frac{a}{4}+z$, whence

$$z^4+a'z^2+b'z+c'=0,$$

where $a'=b-\frac{3}{8}a^2,\quad b'=c-\frac{ab}{2}+\frac{a^3}{8},\quad c'=d-\frac{ac}{4}+\frac{ba^2}{16}-\frac{3a^4}{256}.$

Suppose the roots of this equation are z_1, z_2, z_3, z_4, and that we seek another equation of which the roots are the sums of these taken two and two; then, since $z_1+z_2+z_3+z_4=0$, we have

$(z_1+z_2)=-(z_3+z_4),\quad (z_1+z_3)=-(z_2+z_4),\quad z_1+z_4=-(z_2+z_3).$

Hence, though the required equation would have 6 dimensions, yet since there corresponds to each of its roots another with a contrary sign,

it must be deficient of any terms involving odd powers of the unknown quantity, and it may therefore be reduced to half the dimensions; in other words, its 6 roots are the square roots of the roots of a cubic.

Let $4p$, $4q$, $4r$ be the roots of this cubic; hence

$$z_1+z_2=\pm 2\sqrt{p} \qquad z_3+z_4=\mp 2\sqrt{p}$$
$$z_1+z_3=\pm 2\sqrt{q} \qquad z_2+z_4=\mp 2\sqrt{q}$$
$$z_1+z_4=\pm 2\sqrt{r} \qquad z_2+z_3=\mp 2\sqrt{r}.$$

Add these equations together, observing that $z_1+z_2+z_3+z_4=0$, we will find, on dividing by 2, that

$$z_1=\pm(\sqrt{p}+\sqrt{q}+\sqrt{r})$$

whence $\quad z_2=\mp(\sqrt{p}-\sqrt{q}-\sqrt{r}), \qquad$ since $z_1+z_2=2\sqrt{p}.$

$$z_3=\pm(\sqrt{p}-\sqrt{q}+\sqrt{r})$$
$$z_4=\pm(\sqrt{p}+\sqrt{q}-\sqrt{r}).$$

Such is the reason of the assumption, $z=\sqrt{p}+\sqrt{q}+r$, where p, q, r are the roots of a cubic equation, viz.:—

$$y^3+Ay^2+By+C=0,$$

in which $p+q+r=-A$; $pq+pr+qr=B$; $pqr=-C$.

Squaring the assumed value of z, we have

$$z^2+A=2\sqrt{(pq)}+2\sqrt{(pr)}+2\sqrt{(qr)};$$

and again squaring,

$$(z^2+A)^2-4B=8\sqrt{(pqr)}\,(\sqrt{p}+\sqrt{q}+\sqrt{r})=8\sqrt{(-C)}.z;$$

therefore $\quad z^4+2Az^2-8\sqrt{-C}.z+(A^2-4B)=0;$

whence, by comparison with the given equation in z,

$$A=\frac{a'}{2}; \quad C=-\frac{b'^2}{64}; \quad B=\frac{a'^2}{16}-\frac{c'}{4};$$

the reducing cubic is therefore

$$y^3+\frac{a'}{2}.y^2+\left(\frac{a'^2}{4}-c'\right).\frac{y}{4}-\frac{b'^2}{64};$$

and since $\sqrt{p}\,.\,\sqrt{q}.\,\sqrt{r}=\sqrt{-C}=-\frac{b'}{8}$, the products of these surds are under the condition of having the contrary sign to b', and the other roots are easily found by observing this condition to be

$$z=\sqrt{p}-\sqrt{q}-\sqrt{r}$$
$$z=-\sqrt{p}+\sqrt{q}-\sqrt{r}$$
$$z=-\sqrt{p}-\sqrt{q}+\sqrt{r};$$

and since $x=-\frac{a}{4}+z$, its four values are thus completely determined.

43. We come now to an interesting examination of the quantities influenced by the different surds in the complete expression for the root of the biquadratic, showing first, by *à priori* reflections, what they may be expected to be.

The form of the root, or its simplest type, is

$$x=-\frac{a}{4}+\sqrt{\{P+\sqrt[3]{(Q+\sqrt{R})}+\sqrt[3]{(\ldots\ldots)}\}}+\sqrt{\{P+\ldots\}}$$
$$+\sqrt{\{P+\ldots\}},$$

the blank spaces being occupied with the same quantities Q, R, affected by the quadratic and cubic roots of unity, in the forms already shown in the solution of a cubic.

What do the quantities P, Q, R signify in connexion with the proposed biquadratic? by what formula may they be expressed?

$$\text{Let } \phi \;= x^4+ax^3+bx^2+cx+d.$$
$$\phi' \;= 4x^3+3ax^2+2bx+c.$$
$$\phi'' = 12x^2+6ax+2b.$$
$$\phi'''=24x+6a.$$

And let the equation resulting by the elimination of x between $\phi=0$ and $\phi'=0$ be represented by $[\phi, \phi']$, that between $\phi'=0$ and $\phi''=0$ by $[\phi', \phi'']$, and that between $\phi''=0$ and $\phi'''=0$ by $[\phi'', \phi''']$.

If the proposed equation had two equal roots the reducing cubic would have two equal roots; for the roots of the latter squared are linear functions of the roots of the former, hence the equation $R=0$ and $[\phi, \phi']=0$, of the same degree, imply the same condition, and therefore R can differ from $[\phi, \phi']$ only by a numerical multiplier.

If the biquadratic had three equal roots, the cubic should have the same number of equal roots; the additional conditions necessary for which is, as we have seen, $Q=0$; and Q, like $\sqrt{R}$, is of 6 dimensions.

Now an equation can be formed of six dimensions arranged according to the powers of $\phi''(x)$, which shall be equivalent to the equation $\phi'(x)=0$; that is, which shall express the coexistence of two equal roots. Representing this equation by

$$\{\phi''(x)\}^3+A\{\phi''(x)\}^2+B\phi''(x)+C=0,$$

it is evident that the condition $C=0$ will express the coexistence of three equal roots, and it will be only necessary to calculate this term; we will first show how to form the equation.

$$\text{Let } x_1=x_2, \text{ then } \phi''(x_1)=12x_1^2+6ax_1+b$$
$$=12x_1^2-6x_1(2x_1+x_3+x_4)+2b$$
$$=2b-6x_1(x_3+x_4)$$
$$=2b-6(x_1x_3+x_2x_4).$$

Similarly, if $x_1=x_3$, then $\phi''(x_1)=2b-6(x_1x_2+x_3x_4)$.

Lastly, if $x_2=x_4$, then $\phi''(x_2)=2b-6(x_1x_4+x_2x_3)$;

and it is easy to see conversely, that if

$$\{\phi''(x)-2b+6(x_1x_3+x_2x_4)\}.\{\phi''(x)-2b+6(x_1x_2+x_3x_4)\}.$$
$$\{\phi''(x)-2b+6(x_1x_4+x_2x_3)\}=0,$$

there must exist a pair of equal roots.

If we now put $\phi''(x)=0$, we shall express the same condition as by making $Q=0$, and one of the same dimensions, and we can easily verify the inference that

$$Q=k'\{b-3(x_1x_2+x_3x_4)\}.\{b-3(x_1x_3+x_2x_4)\}.\{b-3(x_1x_4+x_2x_3)\}$$

this product is a symmetrical function of the roots, and may therefore be expressed by the coefficients of the equation; we shall represent it by

$$Q=k'[\phi, \phi', \phi''].$$

The characters between the brackets express the function from which Q is formed; the vanishing of Q being a condition of their co-existence.

For the existence of four equal roots the additional condition $P=0$ is necessary and sufficient; P therefore is of the same dimensions and expresses the same condition as $[\phi'', \phi''']$, when the latter is equated to zero; hence $$P=k''[\phi'', \phi'''].$$

Those who may desire to follow on a more extended scale this mode of considering the solutions of algebraical equations are referred to a separate Memoir which the author has published* on this subject, and in which are given all the factors of the constituent parts of the roots of equations as far as the 5th degree inclusive.

44. The roots of the reducing cubic are functions of the roots of the proposed biquadratic, and we shall terminate our observations on the solution of equations of inferior degrees with the consequences which follow from this consideration.

Let x_1, x_2, x_3, x_4 be the four roots of the biquadratic.

$$x^4+ax^3+bx^2+cx+d=0.$$

That is, $$x_1=-\frac{a}{4}+\sqrt{p}+\sqrt{p}+\sqrt{r}.$$

$$x_2=-\frac{a}{4}+\sqrt{p}-\sqrt{q}-\sqrt{r}.$$

$$x_3=-\frac{a}{4}-\sqrt{p}+\sqrt{q}+\sqrt{r}.$$

$$x_4=-\frac{a}{4}-\sqrt{p}-\sqrt{q}+\sqrt{r}.$$

Hence $$4\sqrt{p}=2x_1+2x_2+a=x_1+x_2-x_3-x_4,$$
$$4\sqrt{q}=2x_1+2x_3+a=x_1+x_3-x_2-x_4$$
$$4\sqrt{r}=2x_1+2x_4+a=x_1+x_4-x_2-x_3.$$

Therefore $$p-q=(\sqrt{p}+\sqrt{q})(\sqrt{p}-\sqrt{q})=\frac{1}{4}(x_1-x_4)(x_2-x_3).$$

$$p-r=(\sqrt{p}+\sqrt{r})(\sqrt{p}-\sqrt{r})=\frac{1}{4}(x_1-x_3)(x_2-x_4).$$

$$q-r=(\sqrt{q}+\sqrt{r})(\sqrt{q}-\sqrt{r})=\frac{1}{4}(x_1-x_2)(x_3-x_4).$$

Now the quantity which is under the final square root in the solution of a cubic is—$(p-q)^2(p-r)^2(q-r)^2$, therefore the same quantity in the biquadratic is

$$-(x_1-x_2)^2(x_1-x_3)^2(x_1-x_4)^2(x_2-x_3)^2(x_2-x_4)^2(x_3-x^4)^2.$$

Thus the quantity under the sign of the final square root in the second, third, and fourth degrees is the product of the squares of the differences of the roots multiplied by constants peculiar to each degree.

In like manner the quantity Q is in the cubic proportional to $(2p-q-r)(2q-p-r)(2r-p-q)$; it follows therefore that the factors of Q in the biquadratic beside a numerical constant are

* Phil. Trans. 1837, Part I.

$$(x_1-x_4)(x_2-x_3)+(x_1-x_3)(x_2-x_4),$$
$$(x_1-x_2)(x_3-x_4)+(x_1-x_4)(x_3-x_2),$$
$$(x_1-x_3)(x_4-x_2)+(x_1-x_2)(x_4-x_3),$$

which agree with those found in the preceding article.

45. The quantity which is under the final square root

$$\kappa z-\frac{1}{12^3}[\phi,\phi']$$

$$=-\frac{1}{12^3}.(x_1-x_2)^2.(x_1-x_3)^2.(x_1-x_4)^2.(x_2-x_3)^2(x_2-x_4)^2(x_3-x_4)^2,$$

where x_1, x_2, x_3, x_4 are the roots of the biquadratic, is now to be discussed.

If all the roots are possible, this quantity is obviously negative, and the solution ceases to be arithmetically practicable.

Suppose two possible, as x_1, x_2, and two impossible, as

$$x_3=m+n\sqrt{-1},\quad x_4=m-n\sqrt{-1}.$$

Then
$$(x_1-x_3)^2(x_1-x_4)^2=\{(m-\alpha)^2+n^2\}^2,$$
$$(x_2-x_3)^2(x_2-x_4)^2=\{(m-\beta)^2+n^2\}^2,$$
$$(x_1-x_2)(x_3-x_4)^2=-4n^2(\alpha-\beta)^2,$$

in which case $-\frac{1}{12^3}[\phi,\phi']$ is positive.

Lastly, if $x_1=m'+n'\sqrt{-1}$, $x_2=m'-n'\sqrt{-1}$, $x_3=m+n\sqrt{-1}$, $x_4=m-n\sqrt{-1}$.

Then
$$(x_1-x_2)^2(x_3-x_4)^2=16n^2n'^2,$$
$$(x_1-x_3)^2(x_2-x_4)^2=\{(m'-m)^2+(n'-n)^2\}^2,$$
$$(x_1-x_4)^2(x_2-x_3)^2=\{(m'-m)^2+(n'+n)^2\}^2;$$

and therefore $-\frac{1}{12^3}[\phi,\phi']$ is here negative.

The numerical solution is thus confined to the case where two roots, and two only, are possible, always excepting the cases of equal roots.

Several writers have fallen into errors with respect to the proper roots of unity which form the coefficients of $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, in the four roots of the biquadratic; the simple rule to avoid error is, "preserve $\sqrt{p}\times\sqrt{p}\times\sqrt{r}$ always of a sign contrary to $8c+a(a^2-4b)$," for this is supposed to be the case in the solution.

It is easy to show that the method of Thomas Simpson leads to the same roots under different forms, and that the reduced cubic has all its roots possible when the biquadratic has only all possible or all impossible roots.

ON THE ROOTS OF UNITY.

Proposition XXI.

46. Two binomial equations of the form $x^a=1$, $y^b=1$, the degrees a, b of which are prime to each other, have no common root beside unity.

For a and b being prime to each other, two other integers, A and B, may always be found such that $aA-bB=\pm 1$;
and the equation $x^a=1$ gives $x^{aA}=1$,
.. .. $y^b=1$ .. $y^{bB}=1$;
therefore, if α be a root common to both, we get by division $\alpha^{(aA-bB)}=1$, whence $\alpha=1^{\pm 1}=1$: thus unity is the only root common to the equations.

Proposition XXII.

47. If α be any root of the equation $x^m=1$, except unity, and p any number intermediate to 1 and m, the latter being supposed a prime number, then α^p shall be a different root of the same equation for all the values of p between the above limits.

First α^p is a root, for α being a root, we have $\alpha^m=1$, therefore $(\alpha^p)^m=1^p=1$.

Secondly, if p and q are different numbers less than m, α^p and α^q are different roots of the equation $x^m=1$; for if not q be the greater number, and $\alpha^q=\alpha^p$; therefore $\alpha^{q-p}=1$, or α is also a root of the equation $y^{q-p}=1$, which is impossible, since $q-p$ is less, and therefore prime to m.

Corollary 1. Hence the $m-1$ different quantities $\alpha, \alpha^2, \alpha^3, \ldots \alpha^{m-1}$ are roots of the equation $x^m=1$, and therefore with unity they contain all its roots, so that the knowledge of any root different from unity is sufficient for the formation of all the other roots.

Corollary 2. Unity is the only real root of the equation $x^m=1$ when m is odd, and $+1$ and -1 are the only real roots when even.

For if $\alpha>1$, then $\alpha, \alpha^2, \alpha^3$, &c. go on increasing, and are therefore all >1; and if $\alpha<1$, then $\alpha, \alpha^2, \alpha^3$, &c. go on decreasing, and must be all <1; therefore, if α be a real quantity different from unity, whatever be its sign, we can never have $\alpha^m=1$, from which the proposition is obvious.

Proposition XXIII.

48. Every rational function of α (where α is an imaginary root of the equation $x^m=1$, and m is prime) is reducible to the form

$$A+B\alpha+C\alpha^2+D\alpha^3+\ldots\ldots P\alpha^{m-1}.$$

For, first, if it contain no function of α as a divisor; that is, if it be an integer function of α, then, arranging it according to the powers of α, any term involving powers higher than the $(m-1)$th powers may be reduced to those which are lower, since $\alpha^m=1$, $\alpha^{m+1}=\alpha$, $\alpha^{m+2}=\alpha^2$, $\ldots\ldots \alpha^{2m}=1$, $\alpha^{2m+1}=\alpha$, &c.; and then the terms which contain like powers being collected will change the form of the function into that stated.

Secondly, if the given function be of a fractional form, as $\frac{F(\alpha)}{\phi(\alpha)}$, where F and ϕ denote integer functions, we can change its form to this equivalent, $\frac{F(\alpha).\phi(\alpha^2).\phi(\alpha^3)\ldots\ldots\phi(\alpha^{m-1})}{\phi(1).\phi(\alpha).\phi(\alpha^2).\phi(\alpha^3)\ldots\ldots\phi(\alpha^{m-1})}.\phi(1)$, the denominator of which is also a symmetrical function of all the roots of the equation $x^m-1=0$, and is simply numerical as well as $\phi(1)$; the expression thus receives an integer form, which, as we have seen, is reducible to that stated.

If the denominator $\phi(a)$ is a simple power of a, as a^n, in which case n may be always made less than m, we may multiply both numerator and denominator be a^{m-n}, and thus the denominator will become unity.

Thirdly, if the function contain several fractions, as above, each being separately reduced to the integer form, it is clear that the aggregate may always be brought to the form given above.

In addition to these properties it is to be remembered, as before mentioned, that the sum of the roots, or of their similar powers, positive or negative, is always zero when the index is not divisible by m, and is m when the index is equal to m or a multiple of it.

Proposition XXIV.

49. The roots of an equation of the form $x^{a.b.c....}=1$ (when a, b, c are different prime numbers) are the terms of the product

$$(1+\alpha+\alpha^2+\ldots\ldots+\alpha^{a-1})\,(1+\beta+\beta^2+\ldots\ldots+\beta^{b-1})$$
$$(1+\gamma+\gamma^2\ \ldots\ldots\ +\gamma^{c-1}),\ \&c.$$

where α is a root of the equation $x^a=1$,
β . . $x^b=1$,
γ . . $x^c=1$,
&c.

and all different from unity.

First, it is clear they are roots of the given equation; for since

$$\alpha^a=1,\ \text{therefore}\ \alpha^{abc\ldots\ldots}=1.$$
$$\text{similarly}\ \beta^{abc\ldots\ldots}=1,$$
$$\gamma^{aba\ldots\ldots}=1,$$
&c.

Therefore the product of α, β, γ, &c., respectively raised to any powers, is obviously a root of the proposed equation.

Secondly, all the roots after the first, which is unity, differ from it, and from each other; for suppose, if it may be possible,

$$\alpha^m.\beta^n.\gamma^p\ldots.=\alpha^{m'}\beta^{n'}\gamma^{p'}\ldots.;\ \text{then}\ \alpha^{m-m'}.\beta^{n-n'}.\gamma^{p-p'}\ldots.=1,$$

all the indices in which may be supposed positive, as they can easily be made so; thus, if $n-n'$ be negative, then $\beta^{n-n'}$ may be put in the form $\beta^{b-(n'-n)}$, where the index is necessarily positive, since n and n' are both less than b.

This being premised, and observing that $\alpha^a=1$, we should have

$$\beta^{n-n'}\gamma^{p-p'}\ldots.=\alpha^{a-(m-m')},$$

the left member of which is a root of the equation $x^{b.c....}=1$, and the right, of the equation $y^a=1$; now these equations *cannot* have a common root, for a, b, c, &c. being prime numbers, a must be prime to $b.c\ldots.$

Since, then, the product mentioned in this proposition contains a, b, $c\ldots.$ terms, all of them roots of the given equation, and all differing from each other, these terms are themselves of necessity all the roots of that equation.

50. *Corollary.* A similar proof evidently applies if a, b, c, &c. are any powers of prime numbers; therefore, if the degree of a binomial equation be any composite number, as μ, we must decompose it into its prime factors $\mu=a^{a'}.b^{b'}.c^{c'}\ldots\ldots$; and having found all the roots of the several equations

$$x^{a^{a'}}=1,\quad y^{b^{b'}}=1,\quad z^{c^{c'}}=1,\ \&c$$

then by taking every possible product of these roots, one selected from each equation, the products thus resulting will be the roots of the equation $x^{\mu}=1$.

Lagrange, and after him other writers, have supposed that the solution of binomial equations with composite indices are always reduced to those with prime indices. This is an error; the reduction has not yet been effected when the index is the power of a prime number: thus, according to Lagrange, if a, being prime, $x^{a^2}=1$, take α, a root of the equation $x^a=1$, and β a root of the equation $y^a=\alpha$, and then all the terms of the product $(1+\alpha+\alpha^2+\ldots\ldots\alpha^{a-1})$ $(1+\beta+\beta^2+\ldots\ldots\beta^{a-1})$ are the roots of the proposed. This is true; but how is β to be found? for since $\alpha=\sqrt[a]{1}$, the equation $y^a=\sqrt[a]{1}$ has for its solution all the difficulties of the proposed.

The same thing can be seen easily by the trigonometrical forms of the roots of unity: thus, if $x^{a.b}=1$, we have

$$x=\cos\left(\frac{m}{ab}\cdot 2\pi\right)+\sqrt{-1}\sin\left(\frac{m}{ab}\cdot 2\pi\right),$$

where 2π denotes the circumference of a circle of which the radius is unity, and m is any number from 0 to $ab-1$ inclusive. Now a, b being prime numbers, the proper fraction $\frac{m}{ab}$ may be decomposed into two proper fractions $\frac{p}{a}+\frac{q}{b}$, and the above value of x will then by trigonometry be the same as

$$x=\left\{\cos\left(\frac{p}{a}\cdot 2\pi\right)+\sqrt{-1}\sin\left(\frac{p}{a}\cdot 2\pi\right)\right.$$
$$\left.\left(\cos\left(\frac{q}{b}.2\pi\right)+\sqrt{-1}\sin\frac{q}{b}\cdot 2\pi\right)\right\},$$

the first and second factors of which are respectively roots of the equations $y^a=1$, $z^b=1$, agreeably with the preceding algebraical theory; but if $a=b$, the fraction $\frac{m}{a^2}$ is undecomposable, except m be a, or a multiple of a; in this case only a of the roots are discoverable by the reduced equation $x^a=1$, the other roots $(a)(a-1)$ in number remaining unknown. This oversight I think it useful to point out, as I am not at present aware that any analyst has attempted the algebraical solution of binomial equations of which the indices are powers of primes.

SOLUTION OF BINOMIAL EQUATIONS OF PRIME DEGREES.

51. It will be convenient to mention those properties of prime numbers on which depends the solution of this class of equations.

If p be a prime number, and a a number prime to p, then $a^{p-1}-1$ will be divisible by p. This theorem was given by Fermat and proved

by Euler; various demonstrations of it are now well known, and may be found in nearly every treatise of the theory of numbers.

The same things supposed, if q be a number $<p-1$, and if a^q-1 be divisible by p, then q is a divisor of $p-1$; when a is such that no numbers of this nature exist, a is called a primitive root of p; there may be several primitive roots of p.

Thus, if $p=$ 3	$a=2$.
.. 5	.. 2, 3.
.. 7	.. 3, 5.
.. 11	.. 2, 6, 7, 8.
.. 13	.. 2, 6, 7, 11.
.. 17	.. 3, 5, 6, 7, 11, 12, 14.
.. 19	.. 2, 3, 10, 13, 14, 15.
.. 23	.. 5, 7, 10, 11, 13, 14, 15, 17, 20, 21.
.. 29	.. 2, 3, 8, 10, 11, 14, 15, 18, 19, 21, 26, 27.
&c.	&c.

But in general the first number in the table may be taken with advantage, being either 2 or 3 in so many cases, and is therefore easily found by trial.

If a is a primitive root of p, then every term of the series $a, a^2, a^3, a^4, \ldots . a^{p-1}$ leave different remainders when divided by p; for suppose that, if possible, a^m and $a^{m'}$ two terms which leave the same remainder, then $a^m-a^{m'}$ is divisible by p, and supposing $m>m'$, we must have $a^{m-m'}-1$ divisible by p, the other factor $a^{m'}$ being prime to p. Now, if this were the case, a would not be a primitive root, contrary to hypothesis.

Proposition XXV.

52. If α be an imaginary root of the equation $x^p=1$, where p is a prime number and a a primitive root to p, then all the roots of the equation are

$$1, \quad \alpha^a, \quad \alpha^{a^2}, \quad \alpha^{a^3}, \quad \alpha^{a^4}, \ldots . \alpha^{a^{(p-1)}}.$$

For let any term of this series have a^m for its index; divide a^m by p, let the quotient be c, and the remainder r; then, since $a^m=cp+r$, therefore $\alpha^{a^m}=(\alpha^p)^c.\alpha^r=\alpha^r$; and we have seen that, in the series of remainders thus arising, r has all integer values from 1 to $p-1$, inclusive, though not in the order of the natural numbers. The above series differs only from the following in the arrangement of the term, viz.: $1, \alpha, \alpha^2, \alpha^3, \alpha^4 \ldots \ldots \alpha^{p-1}$,

which, by the preceding article, are all the roots of the proposed equation. It is easily seen that $\alpha^{a^{p-1}}$ is the same as α.

53. Problem 9. The roots of the equation $x^p=1$ being (besides unity) the series $\alpha, \quad \alpha^a, \quad \alpha^{a^2}, \quad \alpha^{a^3}, \quad \alpha^{a^4}, \quad \ldots . . \alpha^{a^{p-2}}$,

and the roots of the equation $y^{p-1}=1$ being represented by the series

$$1, \omega, \omega^2, \omega^3, \ldots \ldots \omega^{p-2},$$

it is required to find the sum of the products of the corresponding terms in both series.

Let $\quad V=\alpha+\omega.\alpha^a+\omega^2.\alpha^{a^2}+\omega^3.\alpha^{a^3} \ldots \ldots +\omega^{p-2}.\alpha^{a^{p-2}}$,

then, first, V^{p-1} is such a function of the quantities α, α^a, α^{a^2}, &c. as not to change when α^a is put for α; for let V become V_1, when α^a is put for α, we readily see that, since $\alpha = \omega^{p-1}\alpha^{a^{p-1}}$, therefore $V = \omega V_1$; whence $V^{p-1} = V_1^{p-1}$.

And in like manner, if any other two of the same series of quantities be put the one for the other, which is equivalent to substituting some term of the series for α, then V^{p-1} will always remain unaltered.

Now, in raising V to this power, wherever ω receives an index higher than $p-2$ we may depress it, since $\omega^{p-1}=1$, $\omega^{p-2}=\omega$, &c.; hence V^{p-1} is of the form $A_1+\omega A_2+\omega^2 A_3+\ldots\ldots+\omega^{p-2}A_{p-1}$, which being a function that does not change when α^a, α^{a^2}, &c. are put for α, the component functions $A_1, A_2, \ldots A_{p-1}$ must clearly have the same property.

Suppose A_1, A_2, &c. actually expressed in terms of the roots α^a, α^{a^2}, &c. (and it is clear they may also be expressed in terms of the same roots in a different order, viz.: $\alpha, \alpha^2, \ldots\ldots \alpha^{p-1}$); let, therefore,

$$A_1 = a_1 + b_1\alpha + c_1\alpha^a + d_1\alpha^{a^2} \ldots\ldots p_1\alpha^{a^{p-2}}.$$

Put now α^a for α; and since A_1 should not change,

$$A_1 = a_1 + b_1\alpha^a + c_1\alpha^{a^2} + \ldots\ldots + p_1\alpha^{a^{p-1}},$$

whence we see, by comparing both, that $b_1 = c_1$, $c_1 = d_1$, $\ldots p_1 = b_1$;

hence $$A_1 = a_1 + b_1 (\alpha + \alpha^a + \alpha^{a^2} \ldots\ldots \alpha^{a^{p-2}})$$

where a_1 is the term free from α, and b_1 is the coefficient of any power of α below p which we may choose to select, and are therefore known numbers.

Now, since $1+\alpha+\alpha^2+\ldots\ldots\alpha^{p-1}=0$; this gives

$$A_1 = a_1 - b_1;$$

similarly, $$A_2 = a_2 - b_2, \text{ \&c.}$$

hence V^{p-1} is perfectly known, and from thence V_1. Moreover, since $p-1$ is a composite number, we may, by decomposing it into its prime factors, reduce still lower the roots of unity required in the extraction of the value of V.

54. Problem 10. To find all the roots of the equation $x^p=1$, when p is a prime number.

Let $1, \omega_1, \omega_2, \omega_3, \ldots\ldots \omega_{p-2}$ be the roots of the equation $y^{p-1}=1$,

and let
$$V_0 = \alpha + \alpha^a + \alpha^{a^2} + \alpha^{a^3} + \ldots\ldots + \alpha^{a^{p-2}},$$

$$V_1 = \alpha + \omega_1\alpha^a + \omega_1^2\alpha^{a^2} + \omega_1^3\alpha^{a^3} + \ldots\ldots + \omega_1^{p-2}\alpha^{a^{p-2}},$$

$$V_2 = \alpha + \omega_2\alpha^a + \omega_2^2\alpha^{a^2} + \omega_2^3\alpha^{a^3} + \ldots\ldots + \omega_2^{p-2}\alpha^{a^{p-2}},$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$V_{p-2} = \alpha + \omega_{p-2}\alpha^a + \omega_{p-2}^2\alpha^{a^2} + \ldots\ldots + \omega_{p-2}^{p-2}\alpha^{a^{p-2}},$$

where V_0 is manifestly $= -1$, and V_2, V_3 are derived from V_1 by writing ω_2, ω_3, &c. for ω, and the calculation of V_1, as in the last problem, determines all the others.

Then by the properties of the roots of unity already mentioned, we

have
$$
\begin{aligned}
&V_0+V_1+V_2+\quad\ldots\ldots\ldots\quad+V_{p-2}=(p-1)\,a,\\
&V_0+\omega_1^{p-1}V_1+\omega_2^{p-1}V_2+\ldots+\omega_{p-2}^{p-1}V_{p-2}=(p-1)a^{a},\\
&V_0+\omega_1^{p-2}V_1+\omega_2^{p-2}V_2+\ldots+\omega_{p-2}^{p-2}V_{p-2}=(p-1)a^{a^2},\\
&\text{\&c.}\qquad\text{\&c.}
\end{aligned}
$$

Thus all the roots are known.

If, for example, we apply this method to the equation $x^5=1$, we will find

$$4a=-1+\sqrt{5}+\sqrt[4]{(-15+20\sqrt{-1})}+\sqrt[4]{(-15-20\sqrt{-1})}.$$

The rational part in the roots of an equation of any degree is the coefficient of the second term with a changed sign divided by the index of the first; and in the present instance, a, a^2, a^3, a^4, are the roots of the equation $\dfrac{x^5-1}{x-1}=x^4+x^3+x^2+x+1=0$, and accordingly, $-\dfrac{1}{4}$ is the rational part of its roots.

55. This process admits of a simplification, since $p-1$ being a composite number, if c, b, be its prime factors, we may take for ω a root of the equation $z^c=1$ different from unity; this root will manifestly be common to the equation $y^{p-1}=1$.

Then V will be composed of b groups of terms in which the powers of ω will recur in the same order, and which enjoy common properties with V; thus if

$$
\begin{aligned}
U_0&=a+a^{a^c}+a^{a^{2c}}+\ldots\ldots\ldots+a^{a^{(b-1)c}},\\
U_1&=a^{a}+a^{a^{c+1}}+a^{a^{2c+1}}+\ldots\ldots+a^{a^{(b-1)c+1}},\\
U_2&=a^{a^2}+a^{a^{c+2}}+a^{a^{2c+2}}+\ldots\ldots+a^{a^{(b-1)c+2}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
U_{c-1}&=a^{a^{c-1}}+a^{a^{2c-1}}+a^{a^{3c-1}}+\ldots a^{a^{p-2}}.
\end{aligned}
$$

Then V becomes $U_0+\omega U_1+\omega^2U_2+\ldots\ldots\omega^{c-1}U_{c-1}$.

V^c will by the same reasoning be a function not changing when U_0 is changed to U_1, U_1 to U_2, &c., which is precisely the same as changing a into a^a; this is obvious by inspection of the formulæ above given for U_0, U_1, &c., and by adding the c values of V thus deduced, multiplied respectively by the corresponding cth roots of unity U_0, U_1, &c., will obviously be found in a manner strictly analogous to that of the previous process for finding a, a^a, &c.

Next, to deduce from hence the values of a, a^{a^c}, &c., make

$$U=a+\omega' a^{a^c}+\omega'^2 a^{a^{2c}}+\ldots\ldots+\omega'^{b-1}a^{a^{(b-1).c}},$$

where ω' is a root of the equation $u^b=1$; and it is clear by similar reasoning that U^b is a function of a, a^{a^2}, &c., when for a, a^{a^2} … is substituted; U^b will then, like V^{p-1} in the problem before the last, be completely known, and from thence, as in the last problem, a, a^{a^c}, &c. are readily deduced.

Gauss was the first who took advantage of the properties of prime numbers to make the indices of the different roots of unity proceed in geometrical progression, and from that he deduced the solution of binomial equations of prime degrees; the preceding theory, founded on the same idea, and remarkable for simplicity and symmetry, is due to the great analyst, Lagrange.

The same process is easily extended to the cases where $\mu-1$ has several prime factors.

Example 1. Given $x^5-1=0$.

Here $\mu-1=4$, therefore, $c=2$, $b=2$, and ω as well as ω' is a root of the equation $y^2=1$ (which is common to $y^4=1$. The primitive root for 5 is also 2; we have, therefore,

$$\begin{aligned} V &= \alpha+\omega\alpha^2+\omega^2\alpha^{2^2}+\omega^3\alpha^{2^3} \quad (\alpha \text{ being a root of } x^5=1) \\ &= (\alpha+\alpha^4)+\omega(\alpha^2+\alpha^3) \\ &= U_0+\omega U_1 . \end{aligned}$$

Hence, $V^2=(U_0^2+U_1^2)+2\omega U_0U_1$.

Now since $\quad U_0=\alpha+\alpha^4; \quad U_0^2=\alpha^2+\alpha^3+2,$

and since $\quad U_1=\alpha^2+\alpha^3; \quad U_1^2=\alpha^4+\alpha+2,$

therefore, $\quad U_0^2+U_1^2=4+\alpha+\alpha^2+\alpha^3+\alpha^4=3;$

and $\quad U_0U_1=\alpha^3+\alpha+\alpha^4+\alpha^2=-1:$

whence, $\quad V^2=3-2\omega.$

Put for ω its value -1, and form the equations for V_0, V_1.

Hence $\quad V_0=\alpha+\alpha^2+\alpha^{2^2}+\alpha^{2^3}=-1=U_0+U_1;$

$\quad V_1=\alpha+\omega\alpha^2+\omega^2\alpha^{2^2}+\omega^3\alpha^{2^3}=\sqrt{5}=U_0-U_1.$

therefore, $\quad 2U_0=-1+\sqrt{5}, \quad 2U_1=-1-\sqrt{5}.$

Thus, U_0, U_1, are found exactly by the steps indicated in the general process.

Now, to deduce the values of α, α^{2^2}, &c., put as above directed,

$$U=\alpha+\omega'\alpha^{2^2}, \quad \text{where } \omega'=\omega=-1;$$

then $\quad U^2=\left(\alpha+\omega'\alpha^4\right)^2=\alpha^2+\omega'^2\alpha^3+2\omega';$

but $\quad \alpha^2+\alpha^3=U_1=\dfrac{-1-\sqrt{5}}{2}, \quad 2\omega'=-2,$

hence $\quad U^2=\dfrac{-5-\sqrt{5}}{2}$; therefore, $\alpha-\alpha^4=\sqrt{\dfrac{-5-\sqrt{5}}{2}},$

and $\quad \alpha+\alpha^4=U_0=\dfrac{-1+\sqrt{5}}{2}$

whence α, α^4, are known; and similarly α^2, α^3, which differ only in the signs of the radicals.

Example 2. $x^7-1=0$.

Here, $p-1=6$, of which the factors are 2 and 3, and the least primitive root is 3; therefore the roots are of the form

$1, \alpha, \alpha^3, \alpha^{3^2}, \alpha^{3^3}, \alpha^{3^4}, \alpha^{3^5}$, or $1, \alpha, \alpha^3, \alpha^2, \alpha^6, \alpha^4, \alpha^5$, observing that $\alpha^{7m}=1$, when m is integer.

Take for ω a root of the equation $y^2=1$, and therefore ω' is a root of the equation $z^3=1$.

$$\therefore V=\alpha+\omega\alpha^3+\omega^2\alpha^{3^2}+\omega^3\alpha^{3^3}+\omega^4\alpha^{3^4}+\omega^5\alpha^{3^5},$$

$$=(\alpha+\alpha^2+\alpha^4)+\omega(\alpha^3+\alpha^6+\alpha^5),$$

$$=U_0+\omega U_1.$$

Now $\quad U_0^2=(\alpha^2+\alpha^4+\alpha)+2(\alpha^3+\alpha^5+\alpha^6),$

$\quad U_1^2=(\alpha^6+\alpha^5+\alpha^3)+2(\alpha^2+\alpha+\alpha^4),$

therefore $\quad U_0^2+U_1^2=-1-2=-3;$

similarly, $\quad U_0U_1=3-1=2,$

therefore, $\quad V^2=-3+4\omega;$ put now -1 for ω.

Hence, $\quad U_0+U_1=-1, \quad U_0-U_1=\sqrt{-7},$

$$2U_0=-1+\sqrt{-7}, \quad 2U_1=-1-\sqrt{-7}.$$

It remains now to find $\alpha, \alpha^2, \alpha^4$, from U_0, U_1;

put therefore, $\quad U=\alpha+\omega'\alpha^2+\omega'^2\alpha^4,$

then $\quad U^3=A_1+\omega' A_2+\omega'^2 A_3,$

where $\quad A_1=\alpha^3+\alpha^6+\alpha^5+6=U_1+6,$

$\quad A_2=3(\alpha^4+\alpha^2+\alpha)=3U_0,$

$\quad A_3=3(\alpha^6+\alpha^5+\alpha^3)=3U_1.$

Put now for ω' its three values $1, \dfrac{-1+\sqrt{-3}}{2}, \dfrac{-1-\sqrt{-3}}{2}$, and denote, in particular, the middle one, by ω', and let U_0, U_1', U_2' be the corresponding values of U; we have

$$\alpha+\alpha^2+\alpha^4=U_0,$$

$$\alpha+\omega'\alpha^2+\omega'^2\alpha^4=\sqrt[3]{(U_1'^3)},$$

$$\alpha+\omega'^2\alpha^4+\omega'\alpha^4=\sqrt[3]{(U_2'^3)},$$

whence α, &c. are found by addition.

Examples for practice:

$$x^{11}-1=0, \quad x^{13}-1=0, \quad x^{17}-1=0.$$

With respect to the last equation of the seventeenth degree, since $p-1=16=2.2.2.2$ its solution will depend only on quadratic surds, and comparing the real and imaginary parts of the roots with

$$\cos\frac{2m\pi}{17}+\sqrt{-1}\ \sin\frac{2m\pi}{17},$$

where m is any integer below 17, this being the trigonometrical form of the roots, it follows that the division of the circumference of a circle into 17 equal parts depends only on the solution of quadratic surds,

and therefore may be effected by constructions, such as Euclid has given in his fourth book of Elements, for the division of the circumference into 5 equal parts; when $p=5$, then $p-1=4=2.2$; this is the cause of the practicability of the latter by the right line and circle only.

(56.) On the uniform methods proposed for the solution of equations.

It is of great advantage in all branches of analysis to bind together isolated methods of solving problems which are at bottom of the same nature: the advancement of analysis has been principally thus achieved; classification to the chemist and naturalist generate science; comprehensive processes are the perfection of analysis.

Two are here selected of several methods proposed at various times for the solution of equations, all of which in reality have much in common with either that given by Lagrange or that by Bezout, the first distinguished by the symmetry of its processes, and the second by the facility of its application.

Lagrange's method (*Berlin Memoirs*, 1770, 1).

Let the roots of the proposed equation, of the nth degree, be denoted by x_1, x_2, $x_3 \ldots . x_n$, and let ω be one of the imaginary roots of the equation $y^n=1$; lastly, let

$$V=x_1+\omega x_2+\omega^2 x_3+\ldots\ldots+\omega^{n-1}x_n.$$

At first sight the number of values of V^n appears to be $1.2.3\ldots.n$, such being the number of possible permutations of x_1, x_2, $\ldots\ldots x_n$ among each other, but in general they are reducible, for the changes given by these permutations are the same as if the different powers of ω by which they are multiplied were permuted *inter se*.

Now the simultaneous change of x_1 into x_2, x_2 into x_3, x_3 into x_4, x_{n-1} into x_n, and x_n into x_1 evidently changes V into $\omega^{n-1}V$; the simultaneous change of x_1 into x_3, x_2 into x_4, &c. changes V into $\omega^{n-2}V$, and so on; consequently, the changes of V^n are only $1.2.3\ldots.(n-1)$ in number.

Let $$V^n=X_0+\omega X_1+\omega^2 X_2+\ldots\ldots+\omega^{n-1}X_{n-1};$$

the quantities X_0, X_1, X_2, &c. will evidently have the property of not changing when x_1 is changed to x_2, x_2 to x_3, &c. simultaneously; and if we put for ω the successive roots of unity, V^n will take successive values which we may represent by V_0^n, V_1^n, $V_2^n\ldots\ldots V_{n-1}^n$, where

$$\begin{aligned}
V_0 &= x_1+\ x_2+\ \ x_3+\ldots.+x_n\\
V_1 &= x_1+\omega x_2+\omega^2 x_3+\ldots.+\omega^{n-1}x_n,\\
V_2 &= x_1+\omega^2 x_2+\omega^4 x_3+\ldots.+\omega^{2n-2}x_n,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
V_{n-1} &= x_1+\omega^{n-1}x_n+\omega^{n-2}x_3+\ldots.\omega x_n;
\end{aligned}$$

whence x_1, x_2, $\ldots.x_n$, will be known when X_0, $X_1\ldots X_{n-1}$, are known, because these determine V_0, $V_1\ldots.V_{n-1}$.

V_0 is evidently the coefficient of the second term of the equation with the sign changed; and if we add these equations after multiplying each by that power of unity which is supplementary to n, in reference to that by which any root which we wish to find is actually multiplied in these equations, we get

$$nx_1 = V_0 + \quad V_1 + \quad V_2 + \ldots + V_{n-1},$$
$$nx_2 = V_0 + \omega^{n-1} V_1 + \omega^{n-2} V_2 + \ldots + \omega V_{n-1},$$
$$nx_3 = V_0 + \omega^{n-2} V_1 + \omega^{n-4} V_2 + \ldots + \omega^2 V_{n-1},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$nx_n = V_0 + \omega V_1 + \omega^2 V_2 + \ldots\ldots + \omega^{n-1} V_{n-1}.$$

In seeking the quantities V_1^n, $V_2^n, \ldots V_{n-1}^n$, we distinguish the cases of n being a prime or a composite number.

In the first case, suppose these quantities to be roots of the equation $v^{n-1} - av^{n-2} + bv^{n-3} - \&c. = 0$.

Then $$a = V_1^n + V_2^n + \ldots + V_{n-1}^n,$$
$$b = V_1^n V_2^n + V_1^n V_3^n + \ldots + V_{n-2}^n V_{n-1}^n,$$
&c.

the quantities a, b, &c., by the various permutations of x_1 into x_2, &c.; admit of $1.2.3\ldots n-2$ values; the determination of the coefficients of the reduced equation depend therefore on the solution of an equation of this degree, which is higher than the proposed when n is any prime number > 3.

When n is a composite number we can take for ω that root of unity which is common to $y^n = 1$ and $z^p = 1$, p being a factor of n: thus, as in the binomial equations already treated, V will be composed of $\frac{n}{p}$ groups, and so will V^n, by which each of these groups may be determined as before, and then the same process repeated for finding the actual roots, using those roots of unity given when the other factor of n is the index. Of the whole of this general process the preceding solution of binomial equations is an example; a few more will completely illustrate it.

Example. $x^2 - Ax + B = 0$ roots x_1, x_2

In this case ω is a root of the equation $y^2 = 1$ and
$$V = x_1 + \omega x_2$$
$$V^2 = X_0 + \omega X_1$$
$$\text{where } X_0 = x_1^2 + x_2^2 = A^2 - 2B$$
$$X_1 = 2x_1 x_2 = 2B$$

Put for ω its values 1, -1
$$V_0^2 = X_0 + X_1 = A^2$$
$$V_1^2 = X_0 - X_1 = A_2 - 4B$$
and by the general formulæ for the roots applied here
$$2x_1 = V_0 + V_1$$
$$2x_2 = V_0 - V_1$$
(In all cases $V_0 = A$): these are the known values of the roots.

Example 2. $x^3 - Ax^2 + Bx + C$ roots x_1, x_2, x_3.

Here ω is a root of the equation $y^3 - 1 = 0$
$$V = x_1 + \omega x_2 + \omega^2 x_3$$
$$V^3 = X_0 + \omega X_1 + \omega^2 X_2$$
$$X_0 = x_1^3 + x_2^3 + x_3^3 + 6x_1 x_2 x_3$$
$$X_1 = 3(x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1)$$
$$X_2 = 3(x_2^2 x_1 + x_3^2 x_2 + x_1^2 x_3)$$
X_1 X_2 are the roots of a quadratic equation (the coefficients of

which, depending generally on an equation of $1.2\ldots\ldots n-2$ dimensions, are here rational), this equation is easily formed, since X_1+X_2 and X_1X_2 are clearly symmetrical functions of the roots.

$$\begin{aligned}\text{Then } V_0 &= x_1 + x_3\\ V_1 &= x_1 + \omega x_2 + \omega^2 x_3\\ V_2 &= x_1 + \omega^2 x_2 + \omega x_3\end{aligned}$$

give x_1, x_2, x_3, V_1^3, V_2^3, being what V^3 become when $\frac{-1+\sqrt{-3}}{2}$, $\frac{-1-\sqrt{-3}}{2}$ are successively put for unity.

(57.) The method for the general solution of equations given by Bezout in the old Memoirs of the Institute, applies with uniformity and comparative simplicity to several cases, and its processes have been greatly improved by Mr. Lubbock in a MS. memoir, which he has kindly permitted the author to consult; the processes of elimination between several unknown quantities of the first degree, are completely avoided in Mr. Lubbock's method, which is here adopted.

Let $\phi(x)=0$ be an equation of n dimensions of which the roots are sought. Moreover, let $f(y)=0$ be an equation of any given form and of n dimensions in y, and lastly, let $x=F(y)$; the latter function ought to contain n coefficients in order that the result of the elimination of y between $f(y)=0$ and $x-F(y)=0$ may be rendered identical with $\phi(x)$, which has n coefficients; hence $F(y)$ must be at least of $n-1$ dimensions.

Now the equation $f(y)=0$ being of n dimensions, and chosen at pleasure, we may suppose its roots y_1, y_2, $y_3\ldots\ldots y_n$ to be known quantities. (See Art. 36.)

The result of the elimination, which will be the same as $\phi(x)$, if we suppose the coefficient of x^n in the latter to be unity, is therefore

$$\{x-F(y_1)\}\,.\,\{x-F(y_2)\}\,.\,\{x-F(y_3)\}\ldots\ldots\{x-F(y_n)\}=\phi(x)$$

$$\text{Hence Log.}\left\{1-\frac{F(y_1)}{x}\right\}+\text{Log.}\left\{1-\frac{F(y_2)}{x}\right\}+\ldots\ldots$$

$$\text{Log.}\left\{1-\frac{F(y_n)}{x}\right\}=\text{Log.}\frac{\phi(x)}{x^n}$$

in which expression all the logarithms are to be expanded according to the descending powers of x.

$$\text{Now let Log.}\ \frac{\phi x}{x^n}=\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3},\ \&\text{c. } ad\ inf.$$

$$\begin{aligned}\text{and let } F(y_1)+F(y_2)+F(y_3),\ \&\text{c.} &= S_1\\ \{F(y_1)\}^2+\{F(y_2)\}^2+\ \&\text{c.} &= S_2\end{aligned}$$

$$\text{therefore } -\left\{\frac{S_1}{x}+\frac{1}{2}\cdot\frac{S_2}{x^2}+\frac{1}{3}\cdot\frac{S_3}{x^3}+\ \&\text{c.}\right\}=\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3},\ \&\text{c.}$$

comparing the first n terms we find

$$S_1=-A_1\quad S_2=-2A_2\quad S_3=-3A_3\ldots\ldots\quad S_n=-nA_n;$$

that is, we have n equations between the n unknown coefficients which enter the assumed function $F(y)$; if the coefficients be thence deter-

minable, the n values of x (which are the roots sought for the equation $\phi(x)=0$) will be $x=F(y_1)$, $x=F(y_2)\ldots\ldots x=F(y_n)$, where the form of the function $F(y)$ has been determined by the above process.

The principal labour in the application of this method arises in the formation of the symmetrical functions denoted above by S_1, S_2, S_3, &c.; this will obviously be most abridged when we take the function $F(y)$ of the lowest dimensions possible consistent with the number of constants in $\phi(x)$, that is of $n-1$ dimensions; if taken of higher dimensions, some of its terms would remain indeterminate, and we should then be at liberty to impose a certain number of additional conditions on them; but in the case where $f(y)=0$ is the binomial equation, y^n-1, which is chosen generally on account of the simplicity which it gives to the functions S_1, S_2, &c. from the properties of the roots of unity already demonstrated, such additional terms would be utterly useless, since the function of y in this case, of whatever degree it may be, necessarily sinks to one of $(n-1)$ dimensions by Prop. XXIII. For these reasons we shall take $F(y)$ to be of $(n-1)$ dimensions, and $f(y)$ to be y^n-1 in the applications which follow.

Let $F(y)=p+qy+ry^2+sy^3+$, &c.; then putting for y the n roots of unity in this series, and taking the sum, we find $S_1=np=-A_1$, therefore $p=-\dfrac{A_1}{n}$, and it is evident that A_1, which is the first coefficient in the expansion of $\dfrac{\phi(x)}{x^n}$, must be the coefficient of the second term in $\phi(x)$: hence we see, as before, that the rational part of the root of the equation $\phi(x)=0$ is that coefficient with its sign changed and divided by n; we see also that the application of the present method is facilitated by previously taking away the second term, for then $p=0$, and we have only $n-1$ quantities q, r, s, &c. to determine; but no particular advantage arises by taking away more than the second term, for such a change would not alter the number of unknown quantities, and would but little affect the manner in which they are involved.

Example (1.) $\phi(x)=x^2+ax+b=0$

$$f(y)=y^2-1=0 \text{ roots } +1 \text{ and } -1.$$

$$x=p+qy$$

therefore $\phi(x)=\{x-(p+q)\}.\{x-(p-q)\}$

and comparing with the given form we have $2p=-a$, $p^2-q^2=b$

$$\text{therefore } p=-\frac{a}{2} \quad q=\sqrt{\left(\frac{a^2}{4}-b\right)}$$

$$\text{whence } x=-\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}-b\right)}$$

Example (2.) $\phi(x)=x^3+ax+b=0$

$$f(y)=y^3-1=0; \text{ roots } 1,\ \omega,\ \omega^2.$$

$$x=qy+ry^2$$

Therefore,

$$\{x-(q+r)\}\{x-(q\omega+r\omega^2)\}.\{x-(q\omega^2+r\omega)\}=x^3+ax+b$$

$$\text{Log.}\left\{1-\frac{q+r}{x}\right\}+\text{Log.}\left\{1-\frac{q\omega+r\omega^2}{x}\right\}+\text{Log.}\left\{1-\frac{q\omega^2+r\omega}{x}\right\}$$

$$= \text{Log.}\left\{1+\left(\frac{a}{x^2}+\frac{b}{x^3}\right)\right\} = \frac{a}{x^2}+\frac{b}{x^3}-\frac{1}{2}\cdot\frac{a^2}{x^4} \text{ \&c.}$$

therefore $(q+r)^2 + (q\omega+r\omega^2)^2 + (q\omega^2+r\omega)^2 = -2a$

$$(q+r)^3 + (q\omega+r\omega^2)^3 + (q\omega^2+r\omega)^3 = -3b$$

and observing that $1+\omega^2+\omega^4=1$, $1+\omega^3+\omega^6=3$, &c. these equations become

$$3qr = -a$$
$$q^3 + r^3 = -b;$$

from whence we easily find q and r, as in the ordinary solution.

With respect to this method, it may be observed that when the equation $f(y)=0$ is of the binomial form, it no longer essentially differs from Lagrange's method, and when it is not of that form, it will not generally lead to a solution; hence its advantage, if any, consists in the mode of its application.

The problem of the algebraic solution of equations in finite surds consists in reducing a polynomial to a binomial, or system of binomial equations: the last term of the binomial in equations of degrees higher than the fourth is a demi-symmetrical function of the roots of the proposed, which circumstance is due to the fact, that the roots of unity fo a corresponding degree do not enter in a perfectly symmetrical manner in the form of all the roots of the proposed equation except in particular cases easily foreseen; such as in a 5th power we may have $x_1=\sqrt[5]{\alpha}+\sqrt[5]{\beta}$ $x_2=\omega\sqrt[5]{\alpha}+\omega^4\sqrt[5]{\beta}$ $x_3=\omega^2\sqrt[5]{\alpha}+\omega^3\sqrt[5]{\beta}$ $x_4=\omega^3\sqrt[5]{\alpha}+\omega^2\sqrt[5]{\beta}$ $x_5=\omega^4\sqrt[5]{\alpha}+\omega\sqrt[5]{\beta}$ in which ω is used to represent one of the imaginary fifth root of unity; for the perfect symmetry ought to exist not only with respect to ω, ω^2, ω^3, ω^4, ω^5, (by the substitution of any of which for another in the above formulæ we produce another root,) but also with respect to the interchange of α and β (which also happens in this case): now as both these properties cannot exist when the roots of unity have more than two imaginary roots, except in particular cases such as the above, the deficiency of symmetry in the assumed form ought necessarily to cause the reducing binomials to be demi-symmetrical—such too is the actual form obtained in the Memoirs* before quoted.

Since the general form of all the roots in the case above mentioned is included in the equation $x=\omega\sqrt[5]{a}+\omega^4\sqrt[5]{\beta}$, by merely putting for ω all the fifth roots of unity successively, we find by taking the odd powers

$$x^3=(\omega^3 a^{\frac{3}{5}}+\omega^2\beta^{\frac{3}{5}})+3(a\beta)^{\frac{1}{5}}(\omega\sqrt[5]{a}+\omega^4\sqrt[5]{\beta})$$

$$\text{or } x^3-3(a\beta)^{\frac{1}{5}}\,x = \omega^3 a^{\frac{3}{5}}+\omega^2 a^2\beta^{\frac{3}{5}}$$

$$\text{and } x^5=(a+\beta)+5(\omega^3 a^{\frac{3}{5}}+\omega^2\beta^{\frac{3}{5}}).(\alpha\beta)^{\frac{1}{5}}+10(\omega a^{\frac{1}{5}}+\omega^4\beta^{\frac{1}{5}})(a\beta)^{\frac{2}{5}}$$

$$=(a+\beta)+5(a\beta)^{\frac{1}{5}}\{x^3-3(a\beta)^{\frac{1}{5}}x\}+10(a\beta)^{\frac{2}{5}}.x$$

$$\text{therefore } x^5-5(a\beta)^{\frac{1}{5}}.x^3+5(a\beta)^{\frac{2}{5}}x=a+\beta$$

where we see that the particular root of unity (ω) employed disappears, because of the symmetry of the roots when only one pair of the ima ginary roots of unity enters the formula for the roots; and the same

* Transactions of the R. S. 1837.

thing is general for equations of any degree. Hence equations of the form

$$x^5+Ax^3+\tfrac{1}{5}\,.\,A^2x+B=0$$

are easy resolvable.

(58.) We can form the more general equation of which the roots are $x=\omega\sqrt[n]{a}+\omega^{n-1}\sqrt[n]{\beta}$, $x=\omega^2\sqrt[n]{a}+\omega^{n-2}\sqrt[n]{\beta}$, $x=\omega^3\sqrt[n]{a}+\omega^{n-3}\sqrt[n]{\beta}$, &c. (where $\omega^n=1$) in a similar manner, this equation will be of n dimensions, and if we make $\sqrt[n]{\alpha\beta}=p$ we have by Art. 24. Ex. 1.

$$x^n-npx^{n-2}+\frac{n(n-3)}{1.2}.p^2x^{n-4}-\frac{n(n-4)(n-5)}{1.2.3}\,.\,p^3x^{n-6}+\&c.$$
$$=a+\beta$$

Hence an equation of this form may be solved by putting the last term with its sign changed equal to $a+\beta$, and the coefficient of the second term divided by n will give p or $\sqrt[n]{a\beta}$: these equations are sufficient to determine a and β, and from thence all the roots of this equation are known.

But according to the method of Art. 57, we may suppose this equation to arise from the elimination of ω between the equations

$$x=\omega\sqrt[n]{a}+\omega^{n-1}\sqrt[n]{\beta}\text{ and }\omega^n-1=0.$$

Now we may observe, that by the properties of the roots of unity, the sum of any odd power of the different values of x is zero, while in any even power the middle terms repeated n times will alone remain after the addition, all these powers being supposed less than n. Hence

$$S_1=0\quad S_2=2np\quad S_3=0\quad S_4=\frac{4.3}{1.2}.np^2\quad S_5=0\quad S_6=\frac{6.5.4}{1.2.3}p^3\ \&c.$$

which values being substituted in the logarithmic expansion of the last quoted article gives (writing unity for x in that identity).

$$-n\{p+\frac{3}{2}\cdot p^2+\frac{4.5}{2.3}.p^3+\&c.\}$$
$$=\text{Log.}\,\{1-np+\frac{n(n-3)}{1.2}\,.\,p^2-\frac{n(n-4)(n-5)}{1.2.3}\,.\,p^3\,\&c.\}$$

in which equation either series is supposed to be continued until p is raised to the power $\frac{n-1}{2}$ or $\frac{n}{2}-1$; but it is clear that the identity thus true for this number of terms whatever n may be, would be also true were both series continued *ad infinitum*: we thus find that

$$\{1-np+\frac{n(n-3)}{1.2}\cdot p^2-\frac{n(n-4)(n-5)}{1.2.3}p^3+\&c.\}^{-\frac{1}{n}}$$

is a quantity independent of n.

In order to determine this quantity, suppose $n=-1$, and it becomes

$$1+p+\frac{4}{2}\cdot p^2+\frac{5.6}{2.3}\cdot p^3+\frac{6.7.8}{2.3.4}.p^4,\ \&c.$$

$$=\frac{1}{2p}\left\{\frac{1}{2}.4p+\frac{\frac{1}{2}.\frac{1}{2}}{1.2}.(4p)^2+\frac{\frac{1}{2}.\frac{1}{2}.\frac{3}{2}}{1.2.3}.(4p)^3+\frac{\frac{1}{2}.\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{1.2.3.4}.(4p)^4+\&c.\right\}$$
$$=\frac{1-\sqrt{(1-4p)}}{2p}$$

From this result the following remarkable expansions are easily deduced.

$$\left\{\frac{1-\sqrt{(1-4p)}}{2p}\right\}^n = 1+np+\frac{n(n+3)}{1.2}p^2+\frac{n(n+4)\,(n+5)}{1.2.3}p^3+\text{\&c.}$$
ad inf.

$$\left\{\frac{1+\sqrt{(1-4p)}}{2}\right\}^n = 1-np+\frac{n(n-3)}{1.2}.p^2-\frac{n(n-4)\,(n-5)}{1.2.3}.p^3+\text{\&c.}$$
ad inf.

$$\text{Log.}\ \frac{1-\sqrt{1-4p}}{2p} = p+\frac{3}{2}.\,p^2+\frac{4.5}{2.3}.\,p^3+\frac{5.6.7}{2.3.4}.\,p^4+\text{\&c.}$$

(59.) Scholium. There are some important trigonometrical series which are intimately connected with the preceding, and upon which some obscurity is allowed to remain in most treatises on that subject: it will be therefore useful, though somewhat irregular, to consider them here; since simple algebra is sufficient to remove the difficulties which surround them. Whatever may be the value of n, we have generally

$$\left\{\frac{1+\sqrt{(1-4p)}}{2}\right\}^n+\left\{\frac{1-\sqrt{(1-4p)}}{2}\right\}^n$$

$$=1-np+\frac{n(n-3)}{1.2}.\,p^2-\frac{n(n-4)\,(n-5)}{1.2.3}.\,p^3+\text{\&c. } ad\ inf.$$

$$+p^n+np^{n+1}+\frac{n(n+3)}{1.2}.p^{n+2}+\frac{n(n+4)(n+5)}{1.2.3}.\,p^{n+3}+\text{\&c. } ad\ inf.$$

where the general term of the first series is

$$\frac{n(n-m-1)\,(n-m-2)\ldots\ldots(n-2m+1)}{1\ldots\ldots 2\ldots\ldots 3\ldots\ldots\ldots\ldots m}.\,(-p)^m$$

therefore when n is a positive integer the coefficient of $(-p)^m$ will be zero as long as m is between $n-1$ and $\frac{n+1}{2}$ or $\frac{n}{2}+1$ inclusive, namely, $\frac{n+1}{2}$ when n is odd, and $\frac{n}{2}+1$ when n is even; these vanishing terms occur for any integer value of n which is greater than 2: thus if $n=3$ or 4 there is only one vanishing term; if $n=5$ or 6 there are two such terms, and so on. Hence the number of terms of the first series which precede the vanishing terms is $\frac{n+1}{2}$ or $\frac{n}{2}+1$ and the succeeding terms do not again re-appear until m becomes equal to or greater than n. The successive terms which then emerge are exactly equal to the first, second, third, &c. terms of the second series taken with a contrary sign.

Thus the coefficient of

$$p^n=\frac{n.-1.-2.\ldots\ldots-(n-1)}{1\ .\ 2\ .\ 3.\ldots\ldots\ldots\ldots n}.\,(-1)^n=(-1)^{2n-1}=-1$$

the coefficient of

$$p^{n+1}=\frac{n.-2.-3.\ldots\ldots-(n+1)}{1\ .\ 2\ .\ \ 3.\ldots\ldots.(n+1)}\,(-1)^{n+1}=(-1)^{2n+1}.\,n=-n$$

the coefficient of

$$p^{n+2}=\frac{n.-3.-4.\ldots\ldots-(n+3)}{1\,.\,2\,.\,3\ldots\ldots\ldots(n+2)}\,(-1)^{n+2}=(-1)^{2n+3}\,.\,\frac{n\,.\,n+3}{1.2}$$

$$=-\frac{n.n+3}{1\,.\,2}$$

&c. &c.

therefore the terms of the first series which emerge, after the vanishing terms, are destroyed by the addition of the terms of the second series: hence in the case of n being a positive integer we have

$$\left\{\frac{1+\sqrt{(1-4p)}}{2}\right\}^n+\left\{\frac{1-\sqrt{(1-4p)}}{2}\right\}^n=1-np+\frac{n(n-3)}{1\,.\,2}\,.\,p^2-\text{\&c. con-}$$

tinued only for $\frac{n+1}{2}$ terms when n is odd, or $\frac{n}{2}+1$ terms when n is even.

There is another case in which the series is terminating, namely, when n is a negative integer, for then the second series contains vanishing terms, and those which succeed them are destroyed by the first series.

But when n is not an integer, it is obvious that none of the coefficients in either series can vanish, neither can the terms of the second series destroy by addition any terms of the first, since none of the indices of p in the second are then integer, and all the indices in the first are such; we can however introduce the terms of the second in alternate places amongst the first in the order of the increasing magnitude of the indices of p, and the infinite series thus resulting will be the true value of the expansion in this case: thus, let $n=\frac{1}{2}$ then

$$\sqrt{\left\{\frac{1+\sqrt{(1-4p)}}{2}\right\}}+\sqrt{\left\{\frac{1-\sqrt{(1-4p)}}{2}\right\}}$$

$$=1+p^{\frac{1}{2}}+\frac{1}{2}p+\frac{1}{2}p^{\frac{3}{2}}+\frac{1}{2}\frac{\left(\frac{1}{2}-3\right)}{1\,.\,2}\,p^2+\frac{1}{2}\frac{\left(\frac{1}{2}+3\right)}{1\,.\,2}\,p^{\frac{5}{2}}\ \text{\&c.}$$

$$=1+\frac{1}{2}(2p^{\frac{1}{2}})-\frac{1.1}{2.4}.(2p^{\frac{1}{2}})^2+\frac{1.1.3}{2.4.6}.(2p^{\frac{1}{2}})^3+\text{\&c.}$$

$=\sqrt{\{1+2p^{\frac{1}{2}}\}}$ a result which is obviously true.

Having thus obtained clear notions on the algebraical expansion of the sum of the n^{th} powers of the roots of the equation $x^2-x+p=0$, we may deduce the trigonometrical series for the cosine of the multiple arc in terms of the powers of the cosine of the simple arc in the following manner:—

Let $p=\frac{1}{4\cos^2\theta}$ in the preceding equation, of which the roots will then be $\frac{\cos\theta+\sqrt{-1}\sin\theta}{2\cos\theta}$ and $\frac{\cos\theta-\sqrt{-1}\sin\theta}{2\cos\theta}$ the sum of the n^{th} powers of which is $\frac{2\cos n\theta}{(2\cos\theta)^n}$ by Demoivre's theorem, and substi-

tuting in the series above given for the expansion of $\left\{\frac{1+\sqrt{(1-4p)}}{2}\right\}^n$ $+\left\{\frac{1-\sqrt{(1-4p)}}{2}\right\}^n$, and then multiplying both sides by $(2\cos\theta)^n$ we find

$$2\cos n\theta = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{n(n-3)}{1.2}(2\cos\theta)^{n-4} - \&c.$$

$$+ (2\cos\theta)^{-n} + n(2\cos\theta)^{-n-2} + \frac{n(n+3)}{1.2}(2\cos\theta)^{-n-4} + \&c.$$

where if n be a positive or negative integer we are to take only $\frac{n+1}{2}$ or $\frac{n}{2}+1$ terms, of the upper series for the positive value, and of the lower for the negative, inasmuch as all the other terms of either must vanish, or mutually strike out; but when n is not integer, the whole of both series must be retained, but then these series are divergent for real values of θ.

The other expansions for the trigonometrical functions of multiple arcs being deduced from that which represents the cosine, it will be unnecessary to follow them up in this place, having deduced the fundamental theorem applicable to every case, from algebraical principles. For practice the reader may take also the cases of $4p = \frac{1}{\sin^2\theta}$ and $4p = \sin^2(2\theta)$.

(60.) In general, if the monomials of the form assumed for the roots of an equation are not transmutable both with respect to the quantities under the surd signs, and with respect to the roots of unity employed, these surds, on the inverse consideration, are not symmetrical functions of the roots, and cannot therefore be expressed by the coefficients of the equation; this in all cases is necessarily the criterion of the algebraic solubility of equations. The form of expression for those which are reducible to classes of a given form will be facilitated by expansions obtained in that particular form; for instance, the solutions commonly sought are equivalent to the reduction of the equation to the sums of the roots of equations of the form $y^n = a$, $y^n = a'$, $y^n = a''$, &c., but any other sum may, as far as analysis is concerned, be as legitimately adopted; for instance, the form may be the factorial $y(y-h)(y-2h)(y-3h)\ldots.(y-\overline{n-1}.h) = a$; instead then of arranging the left hand member of equations in powers, we should arrange it in functions of the same nature with those of which the inverses are granted to be known; such is the true generalization of the problem of algebraic equations; in symbols it may be represented thus: to reduce the solution of the equation $a\mathrm{F}(x_1 0) + b\mathrm{F}(x_1 1) + c\mathrm{F}(x_1 2) + \ldots\ldots p\mathrm{F}(x_1 n) = 0$, (that is the discovery of the simple relation of $\mathrm{F}(x_1 1)$ and $\mathrm{F}(x_1 0)$) to the conceded solution of the equation $\alpha\mathrm{F}(x_1 0) + \beta\mathrm{F}(x_1 n) = 0$; in the case of factorial equations the following theorem will be useful.

Theorem. Let us denote the product $x(x-h)(x-2h)(x-3h)\ldots.(x-(n-1).h)$ of n factors in arithmetical progression $[x]^n$, then shall

$$[x+y]^n=[x]^n+n[x]^{n-1}y+\frac{n.(n-1)}{1.2}[x]^{n-2}[y]^2.$$
$$+\frac{n(n-1)(n-2)}{1.2.3}.[x]^{n-3}.[y]^3+\&c.$$

a theorem analogous to the binomial and identical with it when $h=0$.

For let z represent any arbitrary quantity, then by the binomial theorem we have

$$(1+z)^{\frac{x}{h}}=1+\frac{x}{h}.z+\frac{x(x-h)}{1.2.h^2}.z^2+\frac{x(x-h)(x-2h)}{1.2.3.h^3}.z^3+\&c.$$

or adopting the notation suggested above for factorials

$$(1+z)^{\frac{x}{h}}=1+x.\frac{z}{h}+\frac{[x]^2}{1.2}.\left(\frac{z}{h}\right)^2+\frac{[x]^3}{1.2.3}.\left(\frac{z}{h}\right)^3+\ldots\ldots+\&c.$$

Similarly

$$(1+z)^{\frac{y}{h}}=1+y.\frac{z}{h}-\frac{[y]^2}{1.2}.\left(\frac{z}{h}\right)^2+\frac{[y]^3}{1.2.3}.\left(\frac{z}{h}\right)^3+\ldots.\&c.$$

Now if the two series are multiplied together, the coefficient of $\left(\frac{z}{h}\right)^n$ in the product is evidently

$$\frac{1}{1.2\ldots.n}\left\{[x]^n+n[x]^{n-1}.y+\frac{n.(n-1)}{1.2}.[x]^{n-2}.y^2+\&c.\right.$$

However, since that product is the same as $(1+z)^{\frac{x+y}{h}}$ it may in like manner be represented by the series

$$1+(x+y).\frac{z}{h}+\frac{[x+y]^2}{1.2}.\left(\frac{z}{h}\right)^2+\frac{[x+y]^3}{1.2.3}.\left(\frac{z}{h}\right)^3+\&c.$$

where the coefficient of the same power $\left(\frac{z}{h}\right)^n$ is $\frac{[x+y]^n}{1.2.3..n}$; the equating this expression with that before obtained gives the identity announced in the theorem.

Example. Let $[x]^2+ax=b$, or $x(x-h)+ax=b$

add to each side $\left[\frac{a}{2}\right]^2$ then

$$[x]^2+ax+\left[\frac{a}{2}\right]^2=b+\left[\frac{a}{2}\right]^2$$

$$\text{or}\quad\left[x+\frac{a}{2}\right]^2=b+\left[\frac{a}{2}\right]^2$$

where $\left[x+\frac{a}{2}\right]^2$ denotes the same as $\left(x+\frac{a}{2}\right)\left(x+\frac{a}{2}-h\right)$;

such in this case is the simple equation to which the proposed quadratic is reducible.

Thus it will be seen that the solution of equations in the algebraical sense only consists in reducing them to binomials of a particular form, and that form has the advantage which contains only pure powers of

the unknown quantity; but the question admits of extension to any form of function in which x may be regularly involved.

(61.) The early analysts (more particularly Tchirnhausen) have been much occupied with methods for taking away the coefficients of equations, and the same track has been pursued recently by Mr. Jerrard in his "Researches," with a power of notation much called for in the complicated involutions of the higher degrees, and most creditable to the inventor: but towards the solution of equations by the reduction of polynomial to binomial equations—these or any other proposed methods cannot advance nearer than the general methods of Bezout (*Mémoires de l'Acad. des Sciences*, 1765). Nevertheless an extended and close examination of the properties of what may be called conjugate equations, that is, those mutually reducible, or having analogous relations to the analytical *reduced* equation, would have more value than as merely speculations. Like the properties of elliptic functions, which, though not reducible to circular or logarithmic, have reducible differences, and have most useful applications in the pure and physical mathematics: so further researches into the surd transcendants, which constitute the roots of equations of the higher degrees, may not improbably remove some of the existing difficulties in differential equations, and this surmise is only introduced to show that those who have reaped less fruit from this class of researches than they have bestowed labour, ought to turn their attention to extract valuable results, though different from the object for which they originally started. The reader who will consult a Memoir presented by the Author to the Royal Society, on the analysis of the roots of equations, will see that many beautiful properties are couched in the surd expression alone of the roots.

(62.) On the Solution of Equations by Series.

The method for obtaining series for the roots of equations contained in this article is taken from a Memoir communicated by the Author to the Cambridge Philosophical Society, and may be found by referring to the fourth volume of the Transactions of that body.

Let the given equation be arranged according to the powers of the unknown quantity x, and by division make the coefficient of the first power of x to be unity.

Divide then the equation by x, and take the Naperian logarithm of the quotient of its left member by the known formula

$$\text{Log. } (1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\&c.$$

Take the coefficient of $\frac{1}{x}$ in this series, seeking what it is in each term; this with its sign changed will be a root of the equation.

For let the equation be $\phi(x)=0$, and its roots α, β, γ, &c., then

$$\phi(x)=\text{A}(x-\alpha)(x-\beta)(x-\gamma)(\ldots\ldots\&c.$$

where A is independent of x.

$$\text{Hence}\quad \phi(x)=\text{A}'(x-\alpha)\left(1-\frac{x}{\beta}\right)\left(1-\frac{x}{\gamma}\right)\ldots$$

where $\text{A}'=\text{A}\,.\,(-\beta)\,(-\gamma)\ldots\ldots$

$$\text{Therefore}\quad \frac{\phi(x)}{x}=\text{A}'\left(1-\frac{\alpha}{x}\right)\left(1-\frac{x}{\beta}\right)\left(1-\frac{x}{\gamma}\right)\ldots\ldots$$

and $\text{Log.}\ \frac{\phi(x)}{x} = \text{Log. A}' + \text{Log.}\left(1 - \frac{\alpha}{x}\right) + \text{Log.}\left(1 - \frac{x}{\beta}\right)$

$$+ \text{Log.}\left(1 - \frac{x}{\gamma}\right) + \ldots\ldots$$

The only term in the right hand member which contains negative powers of x is manifestly

$$\text{Log.}\left(1 - \frac{\alpha}{x}\right) = -\frac{\alpha}{x} - \frac{1}{2} \cdot \frac{\alpha^2}{x^2} - \frac{1}{3} \cdot \frac{\alpha^3}{x^3} + \&\text{c.}$$

The coefficient of $\frac{1}{x}$ therefore in $\text{Log.}\ \frac{\phi(x)}{x}$ is $-\alpha$, which with a changed sign is a root of the equation.

(Example 1.) Given $x^2 + ax + b = 0$ to find a root α.

$$\text{Here } \frac{\phi(x)}{x} = a + x + \frac{b}{x}$$

$$\text{Log.}\left(\frac{\phi(x)}{x}\right) = \text{Log.}\ (a) + \text{Log.}\ (1+z) \text{ where } z = \frac{1}{a}\left(x + \frac{b}{x}\right)$$

Hence $-\alpha =$ coefficient of $\frac{1}{x}$ in $z - \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4}$ &c.

Now the coefficient of $\frac{1}{x}$ in $z = \frac{b}{a}$

$$\ldots\ldots\ldots\ldots \text{ in } z^3 = \frac{3b^2}{a^3}$$

$$\ldots\ldots\ldots\ldots \text{ in } z^5 = \frac{10b^3}{a^5}$$

$$\&\text{c.} = \&\text{c.}$$

and no term involving $\frac{1}{x}$ appears in the even powers of z.

$$\text{Hence } \alpha = -\left(\frac{b}{a} + \frac{b^2}{a^3} + \frac{2b^3}{a^5} + \frac{5b^4}{a^7} + \frac{14b^5}{a^9} + \&\text{c.}\right)$$

We can easily obtain, if we desire it, the general term of the series; thus, since $z^{2n+1} = \left(x + \frac{b}{x}\right)^{2n+1}$

$$\text{Coefficient of } \frac{1}{x} \text{ in } \frac{z^{2n+1}}{2n+1} = \frac{(2n-1)\,(2n-2)\ldots\ldots(n+2)}{2\,.\,3\qquad 4\,.\,.\,.\,.\,.\,n} \cdot \frac{b^{n+1}}{a^{2n+1}}$$

$$\text{Thus } \alpha = -\left\{\frac{b}{a} + \frac{b^2}{a^3} + \frac{4}{2} \cdot \frac{b^3}{a^5} + \frac{6.5}{2.3} \cdot \frac{b^4}{a^7} + \frac{8.7.6}{2.3.4} \cdot \frac{b^5}{a^9} + \&\text{c.}\right)$$

We may observe that one root found by the surd solution is

$$x = -\frac{a}{2} + \sqrt{\left(\frac{a^2}{4} - b\right)}$$

$$= -\frac{a}{2} + \frac{a}{2}\left(1 - \frac{4b}{a^2}\right)^{\frac{1}{2}}$$

$$= -\frac{a}{2}\left\{\frac{1}{2}\cdot\frac{4b}{a^2}+\frac{1.1}{2.4}\cdot\left(\frac{4b}{a^2}\right)^2+\frac{1.1.3}{2.4.6}\left(\frac{4b}{a^2}\right)^3+\&c.\right\}$$

which expression is obviously identical with that above obtained.

Let S_n be the nth term of either series, it is evident that

$$S_n = -\frac{2n(2n-1)\ (2n-2)\ldots\ldots(n+2)}{2\ .\ \ 3\ .\ \ \ 4\ .\ .\ .\ .\ n}\cdot\frac{b^{n+1}}{a^{2n+1}}$$

and $$S_{n-1} = -\frac{(2n-2)\ (2n-3)\ldots\ldots(n+1)}{2\ \ .\ \ 3\ .\ .\ .\ .\ n-1}\cdot\frac{b^n}{a^{2n-1}}$$

therefore $$S_n = \frac{2n(2n-1)}{(n+1).n}\cdot\frac{b}{a^2}.S_{n-1}$$

$$= \frac{1-\dfrac{1}{2n}}{1+\dfrac{1}{n}}\cdot\frac{4b}{a^2}.S_{n-1}$$

When n is sufficiently great, if $4b>a^2$ S_n will be $>S_{n-1}$ and therefore the series must in such case become divergent as the roots of the equation become imaginary.

Before leaving this example, we shall examine which root of the equation the series gives when convergent; let α, β, be the two roots,

$$\alpha+\beta=-a \qquad \alpha\beta=b \quad \text{hence}$$

$$\text{Root} = \frac{\alpha\beta}{\alpha+\beta}+\frac{\alpha^2\beta^2}{(\alpha+\beta)^3}+\frac{2\ \alpha^3\beta^3}{(\alpha+\beta)^5}+\&c.$$

Let α be the least, then

$$\frac{\alpha\beta}{\beta+\alpha}=\alpha\beta\left(\beta+\alpha\right)^{-1}=\alpha\left(1-\frac{\alpha}{\beta}+\frac{\alpha^2}{\beta^2}-\frac{\alpha^3}{\beta^3}\ \&c.\ ad\ inf.\right)$$

because the series between brackets being convergent, we may regard it as strictly true when continued *ad inf.*

Again $$\frac{\alpha^2\beta^2}{(\beta+\alpha)^3}=\alpha.\frac{\alpha}{\beta}\left(1+\frac{\alpha}{\beta}\right)^{-3}=\alpha\left(\frac{\alpha}{\beta}-3\frac{\alpha^2}{\beta^2}+6\frac{\alpha^3}{\beta^3}\ \&c.\right.$$

$$\frac{2\,\alpha^3\beta^3}{(\alpha+\beta)^5}=\alpha.\frac{\alpha^2}{\beta^2}\left(1+\frac{\alpha}{\beta}\right)^{-5}=2\,\alpha\left(\frac{\alpha^2}{\beta^2}-5\frac{\alpha}{\beta}\ \&c.\right)$$

hence if we add, all the terms mutually destroy except the first, and therefore the series represents strictly the *least* root.

This series is therefore a *discontinuous* function of α,β: being rational, it can only represent one of them at a time, though both are analytically involved in the same manner. If we suppose one of them, as α, to be variable, and the second, β, to be constant, the variable will be truly represented by the series, when it is less than the constant, and the constant afterwards will be represented. For further information on this subject, consult a Memoir by the Author, in Vol. IV., Camb. Trans., with the title, 'First Memoir on the Inverse Method of Definite Integrals.'

If the proposed equation $\phi(x)=0$ contains no term involving the first power of x, we may put $x=z+h$, and seek z by the same method,

or write the equation in the form $x+\phi(x)-x=0$ and expand the Log. $\left\{1+\frac{\phi(x)-x}{x}\right\}$ and take the coefficient of $\frac{1}{x}$.

Example (2). Given $x^n+ax+b=0$

$$\text{Here } \frac{\phi(x)}{ax}=1+\frac{1}{ax}(b+x^n)=1+\frac{z}{ax}$$

suppose. Therefore

$$\text{Root} = \text{coefficient of } \frac{1}{x} \text{ in } -\text{Log.}\left(1+\frac{z}{ax}\right)$$

$$= \ldots\ldots \text{ in } -\frac{z}{ax}+\frac{1}{2}\cdot\frac{z^2}{a^2x^2}-\frac{1}{3}\cdot\frac{z^3}{a^3x^3} \quad \&c.$$

that is, we must take the absolute term in $-\frac{z}{a}$, the coefficient of x in $\frac{1}{2}\cdot\frac{z^2}{a^2}$, the coefficient of x^2 in $-\frac{1}{3}\cdot\frac{z^3}{a^3}$ &c.

Now since $z=b+x^n$ it is clear that the only powers of x which enter z, z^2, z^3, &c., are the nth, $2n$th, &c.; hence, besides the absolute term in $-\frac{z}{a}$, we have the coefficient of x^n in $(-1)^{n+1}\cdot\frac{1}{n+1}\cdot\frac{z^{n+1}}{a^{n+1}}$ of x^{2n} in $-\frac{1}{2n+1}\cdot\frac{z^{2n+1}}{a^{2n+1}}$ &c.; these are easily found, and collecting them, we obtain

$$\alpha=-\frac{b}{a}-\frac{b^n}{(-a)^{n+1}}-\frac{2n\,b^{2n-1}}{2.\,a^{2n+1}}-\frac{3n(3n-1)b^{3n-2}}{2.\,3\,(-a)^{3n+1}}-\&c.$$

which includes the former example when $n=2$.

Example (3). To find α a root of the equation $x=\varepsilon^{ax}$

We have here $\phi(x)=x-\varepsilon^{ax}$

therefore, α = coefficient of $\frac{1}{x}$ in $-\text{Log.}\left(1-\frac{\varepsilon^{ax}}{x}\right)$

ε being the base of Naperian logarithms.

$$\text{Hence } \alpha = \text{coefficient of } \frac{1}{x} \text{ in } \left\{\frac{\varepsilon^{ax}}{x}+\frac{1}{2}\cdot\frac{\varepsilon^{2ax}}{x^2}+\cdot\frac{\varepsilon^{3ax}}{3x^3} \quad \&c.\right\}$$

Expand the exponential functions by the formula,

$$\varepsilon^z=1+z+\frac{z^2}{1.2}+\frac{z^3}{1.2.3} \quad \&c.$$

$$\text{Whence } \alpha=1+\frac{2a}{1.2}+\frac{3^2a^2}{1.2.3}+\frac{4^3a^3}{1.2.3.4}+ \quad \&c.$$

a series of which we shall examine the divergence or convergence.

Represent the nth term by s_n, we have

$$s_n=\frac{(n\,a)^{n-1}}{1.2.3\ldots(n-1).n} \qquad s_{n+1}=\frac{(n+1)^n a^n}{1.2.3\ldots n.(n+1)}$$

same in $-\left\{\frac{1}{a}\left(x+\frac{b}{x^{n-1}}\right)-\frac{1}{2a^2}\left(x+\frac{b}{x^{n-1}}\right)^2+\frac{1}{3a^3}\left(x+\frac{b}{x^{n-1}}\right)^3, \text{\&c.}\right\}$; and it is obvious that the quantity $\frac{1}{x}$ appears only in the $(n-1)^{\text{th}}$ and succeeding terms. This sum is therefore the coefficient of $\frac{1}{x}$ in the following series, viz.

$$(-1)^{n-1}\left\{\frac{1}{(n-1)\,a^{n-1}}\left(x+\frac{b}{x^{n-1}}\right)^{n-1}+\frac{(-1)^n}{(2n-1)\,a^{2n-1}}\left(x+\frac{b}{x^{n-1}}\right)^{2n-1}+\text{\&c.}\right\}.$$

Its value therefore is—

$$\frac{b}{(-a)^{n-1}}-\frac{2n-2}{2}\cdot\frac{b^2}{(-a)^{2n-1}}+\frac{(3n-2)(3n-3)}{2.3}\cdot\frac{b^3}{(-a)^{3n-1}}-\text{\&c.}$$

When $x=2$ we get the root of the quadratic before found. When $x=3$ it gives $\frac{b}{a^2}-\frac{2b^2}{a^5}+\frac{7b^3}{a^8}-$&c. as the sum of two roots of the cubic $x^3+ax^2+b=0$; and therefore the third root is $-a-\frac{b}{a^2}+\frac{2b^2}{a^5}-\frac{7b^3}{a^8}+$&c.

By this theorem we can find all the roots of the equation $\phi(x)=0$; for we have only to subtract the sum of $m-1$ roots from that of m to find the mth. The reader must refer to the Memoirs already mentioned to see that this method gives the m (numerically) least roots, not considering the sign.

Example 3. $x^m=\varepsilon^{ax}$.

Here the required sum is the coefficient of $\frac{1}{x}$ in $-\text{Log.}\left(1-\frac{\varepsilon^{ax}}{x^m}\right)$, the value of which, it is easily seen, is—

$$\frac{a^{m-1}}{1.2\ldots(m-1)}+\frac{(2a)^{2m-1}}{1.2\ldots(2m-1)}+\frac{(3a)^{3m-1}}{1.2\ldots(3m-1)}, \text{\&c.}$$

The sum of $m-1$ roots is got by putting the equation under the form

$$x^{m-1}=\varepsilon^{\left(a-\frac{a}{m}\right)x}$$

and repeating this process.

(64.) To find any given rational and integer function of a root of an algebraical equation.

Let $\phi(x)=0$ be the proposed equation, of which suppose the roots to be α, β, γ, &c.; then, C denoting a quantity independent of x, we must have—

$$\phi(x)=\text{C}\,(x-\alpha)(x-\beta)(x-\gamma)\;\ldots\ldots$$

$$\text{and } \frac{\phi(x)}{x}=\text{C}'\left(1-\frac{\alpha}{x}\right)\left(1-\frac{x}{\beta}\right)\left(1-\frac{x}{\gamma}\right)\ldots\ldots$$

$$\text{therefore Log. }\frac{\phi(x)}{x}=\text{Log. C}'+\text{Log.}\left(1-\frac{\alpha}{x}\right)+\text{Log.}\left(1-\frac{x}{\beta}\right)$$

$$+\text{Log.}\left(1-\frac{x}{\gamma}\right)+\ldots\ldots=\text{Log. C}'-\left(\frac{\alpha}{x}+\frac{1}{2}\cdot\frac{\alpha^2}{x^2}+\frac{1}{3}\cdot\frac{\alpha^3}{x^3}+\ldots\right)$$

$$-x\left(\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}+\ldots\right)-\frac{x^2}{2}\left(\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}+\ldots\right)$$

$-$&c. &c.;

in which expansion it is visible that the negative powers of x are multiplied by the positive powers of α, and the multiplier of $\frac{1}{x^n}$ is $-\frac{\alpha^n}{n}$; in other words, α^n is the coefficient of $\frac{1}{x}$ in the expansion of the function $-n x^{n-1}$ Log. $\frac{\phi(x)}{x}$; and, making an *accented* quantity represent the derived function of that quantity relative to x, we find a^n = coefficient of $\frac{1}{x}$ in $-(x^n)'$ Log. $\frac{\phi(x)}{x}$.

Hence, if $f(\alpha)$ be any function consisting of the positive and integer powers of α, *excluding* the absolute term, it follows that

$$f(\alpha) = \text{coefficient of } \frac{1}{x} \text{ in } -f'(x) \text{ Log. } \frac{\phi(x)}{x};$$

for this is only to take n successively equal to 1, 2, 3, &c., in the formula above found, and, multiplying by the proper coefficients in the expansion of $f(x)$, to add together the results.

Example 1. What series represents the square of the least root of the equation $x^2+ax+b=0$?

The given function being x^2, the derived function is $2x$, and therefore α^2 = coefficient of $\frac{1}{x}$ in $-2x$ Log. $\left\{1+\frac{1}{a}\left(x+\frac{b}{x}\right)\right\}$; that is, in $-2x\left\{\frac{1}{a}\left(x+\frac{b}{x}\right)-\frac{1}{2a^2}\left(x+\frac{b}{x}\right)^2+\frac{1}{3a^3}\left(x+\frac{b}{x}\right)^3-\&c.\right\}$; hence, in the development between the brackets we must take the coefficient of $\frac{1}{x^2}$ in the second, fourth, sixth, &c. terms, and multiply their sum by -2. We thus find the required series to be

$$\frac{b^2}{a^2}+\frac{2b^3}{a^4}+\frac{5b^4}{a^6}+\&c.$$

For the coefficient of $\frac{1}{x^2}$, in the $(2n)$th term of the above-written series, is $-\frac{b^{n+1}}{2na^{2n}}\cdot\frac{2n(2n-1)\ldots\ldots(n)}{1.2\ \ldots\ldots n+1}$, and therefore the corresponding term of the required series is $\frac{2b^{n+1}}{a^{2n}}\cdot\frac{(n)(n+1)\ldots\ldots(2n-1)}{2.3\ldots\ldots n+1}$.

Example 2. To find the value of x^n in the equation $x=c\varepsilon^{ax}$.

Since $-\text{Log.}\left(1-\frac{c\varepsilon^{ax}}{x}\right)=\frac{c\varepsilon^{ax}}{x}+\frac{1}{2}\frac{c^2\varepsilon^{2ax}}{x^2}+\frac{1}{3}\frac{c^3\varepsilon^{3ax}}{x^3}+\&c.$, therefore we must select in this case the coefficient of $\frac{1}{x}$ from the terms of the series

$$n\left\{\frac{1}{n}\cdot\frac{c^n\varepsilon^{nax}}{x}+\frac{1}{n+1}\cdot\frac{c^{n+1}\varepsilon^{(n+1)ax}}{x^2}+\frac{1}{n+2}\cdot\frac{c^{n+2}\varepsilon^{(n+2)ax}}{x^3}, \&c.\right\},$$

which is easily found to be

$$c^n+nac^{n+1}+\frac{n}{2}(n+2)\,a^2c^{n+2}+\frac{n}{2}\cdot\frac{(n+3)^2}{3}\cdot a^3c^{n+3}+\text{\&c.}$$

If, instead of dividing $\phi(x)$ by x, we divide by x^m, it is plain that the same method shows that the sum of one and the same function of m of roots is found by subtracting the coefficient of $\frac{1}{x}$ in the expansion of $f'(x)\,\text{Log.}\,\frac{\phi(x)}{x^m}$ from $mf(0)$.

By subtracting the sum so found for m roots from that corresponding to $m+1$ roots, we may obtain successively the given function of each root of the equation.

When this function is not developable in positive integer powers of x, we may apply the same method by making $x=k+z$, leaving k arbitrary, and considering the transformed equation as one in which z is the unknown quantity.

We must have recourse to the same expedient when there is no term in the proposed equation with the same exponent as that power of x by which we are to divide; for the logarithmic expansion in x then fails.

(65.) Problem. If $\phi(y)$ represent any given function of y, it is required to expand according to the powers of k any other given rational and entire function of y, as $f(y)$, y being connected with k by the equation

$$y=a+k\phi(y).$$

Put $y=x+a$, and this equation becomes

$$x-k\phi(a+x)=0\,;$$

and we are to find the expansion of $f(y)$, or $f(a+x)$.

The principles explained in the preceding articles give

$$f(y)-f(a)=\text{coefficient of }\frac{1}{x}\text{ in }-f'(a+x)\,\text{Log.}\left\{1-\frac{k}{x}\phi(a+x)\right\},$$

where $f(a)$ is subtracted, being the absolute term.

Let $\psi_1(a)$ be put for abridgment for $f'(a)\cdot\phi(a)$, $\psi_2(a)$ for $f'(a).(\phi a)^2$, &c., hence $f(y)=f(a)+$ the coefficient of $\frac{1}{x}$ in the following series, viz.

$$\left\{\frac{k}{x}\psi_1(a+x)+\tfrac{1}{2}\cdot\frac{k^2}{x^2}\cdot\psi_2(a+x)+\tfrac{1}{3}\cdot\frac{k^3}{x^3}\cdot\psi_3(a+x),\ \text{\&c.}\right\},$$

and, expanding the functions by the general formula

$$\psi(a+x)=\psi(a)+x\psi'(a)+\frac{x^2}{1.2}\cdot\psi''(a)+\text{\&c.},$$

we find

$$f(y)=f(a)+k\psi_1(a)+\frac{k^2}{2}\cdot\{\psi_2(a)\}'+\frac{k^3}{2.3}\cdot\{\psi_3(a)\}'',\ \text{\&c.}$$

Corollary. When $f(y)=y$, then $f'(a)=1$, and $\psi_1(a)=\phi(a)$, $\psi_2(a)=\{\phi(a)\}^2$, &c., and therefore

$$y=a+k\phi(a)+\frac{k^2}{2}\cdot\{(\phi(a)^2\}'+\frac{k^3}{2.3}\cdot\{(\phi a)^3\}'',\ \text{\&c.}$$

These theorems are due to Lagrange, but were obtained by him from the calculus of derived functions.

For other similar but more extensive theorems, as comprehending several roots of the equation, it will be here sufficient to refer to the Memoir, already mentioned, on the solution of algebraical equations, in the Cambridge Philosophical Transactions.

A remarkable inference relative to series of the above form is that, if we multiply two of them, the product will be of a similar form; and so will the quotient when one of them is divided by the other, the fundamental equation $y=a+k\phi(y)$ being common to all. For, if $F(y)$ be another function of y, and $\Pi(y)=f(y).F(y)$, we have simultaneously

$$f(y)=f(a)+kf'(a).\phi(a)+\frac{k^2}{2.3}.\{f'(a).(\phi a)^2\}'$$
$$+\frac{k^3}{2.3.4}.\{f'(a).(\phi a)^3\}''+\&c.$$

$$F(y)=F(a)+kF'(a).\phi(a)+\frac{k^2}{2.3}.\{F'(a).(\phi a)^2\}'$$
$$+\frac{k^3}{2.3.4}\{F'(a).(\phi a)^3\}''+\&c.$$

$$\Pi(y)=\Pi(a)+k\Pi'(a).\phi(a)+\frac{k^2}{2.3}.\{\Pi'(a).(\phi a)^2\}$$
$$+\frac{k^3}{2.3.4}.\{\Pi'(a)(\phi a)^3\}''+\&c.$$

The third series is therefore the product of the first and second, and the second is the quotient of the first divided by the third. This property may be easily verified by actual multiplication.

(66.) Problem XII. To find the sum of the inverse nth powers of the roots of the equation $x=a+h\phi(x)$.

Let α, β, γ, &c. be the roots of the equation; then, C denoting a quantity independent of x, and $\phi(x)$ being supposed rational and integer, we have

$$a-x+h\phi(x)=C(x-\alpha)(x-\beta)(x-\gamma)\ldots.$$

and $$\text{Log.}\,(a-x)+\text{Log.}\left\{1+\frac{h\phi(x)}{a-x}\right\}=\text{Log. }C'+\text{Log.}\left(1-\frac{x}{\alpha}\right)$$
$$+\text{Log.}\left(1-\frac{x}{\beta}\right)+\ldots.$$

where $C'=C(-\alpha)(-\beta)(-\gamma)\ldots.$

Now $$\text{Log.}\left(1-\frac{x}{\alpha}\right)+\text{Log.}\left(1-\frac{x}{\beta}\right)+\ldots.=-x\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\ldots.\right)$$
$$-\tfrac{1}{2}x^2\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\ldots.\right)$$
$$-\tfrac{1}{3}x^3\left(\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\ldots.\right)$$

Therefore $\frac{1}{\alpha^n}+\frac{1}{\beta^n}+\frac{1}{\gamma^n}+\ldots.$ is the coefficient of x^n in the expansion

of $-n\,\text{Log.}\left(1-\frac{x}{a}\right)-n\,\text{Log.}\left\{1+h\frac{\phi(x)}{a-x}\right\}$

$$=\frac{1}{a^n}-\text{coefficient of } x^n \text{ in } -\frac{nh}{a-x}\phi(x)+\frac{nh^2}{2(a-x)^2}(\phi x)^2$$

$$-\frac{nh^3}{3(a-x)^3}.(\phi x)^3-\&c.$$

Let $\phi(x)=A_0+A_1x+A_2x^2+A_3x^3+\ldots+A_nx^n+\&c.$;

then, since $\frac{1}{a-x}=\frac{1}{a}+\frac{x}{a^2}+\frac{x^2}{a^3}+\frac{x^3}{a^4}+\ldots+\frac{x^n}{a^{n+1}}+\&c$,

therefore the coefficient of x^n in $\frac{\phi(x)}{a-x}=\frac{A_0}{a^{n+1}}+\frac{A_1}{a^n}+\frac{A_2}{a^{n-1}}+\ldots$

$$+\frac{A_n}{a}=\frac{1}{a^{n+1}}\phi(a),$$

provided we admit none but negative powers of a.

Let $f(a)=\frac{1}{a^n}$, then $f'(a)=-\frac{n}{a^{n+1}}$.

Hence the coefficient of x^n in $\frac{-nh\phi\,(x)}{a-x}=hf'\,(a).\phi\,(a)$ under the restriction above mentioned.

For the same reason, the coefficient of x^n in

$$\frac{-nh^2\,(\phi x)^2}{a-x}=h^2 f'\,(a)\,(\phi\,(a))^2$$

and taking the derived functions relative to a, we have the coefficient of x^n in $\frac{nh^2\,(\phi x)^2}{(a-x)^2}=h^2\left\{f'\,(a).(\phi a)^2\right\}'$, recollecting that only negative powers of a are retained.

Write $(\phi\,(x))^3$ for $(\phi x)^3$, and take the derived functions relative to a, hence the coefficient of x^n in $\frac{-nh^3\,(\phi x)^3}{(a-x)^3}=\frac{h^3}{2}\left\{f'\,(a).(\phi a)^3\right\}''$ and so on for the remaining terms.

Substituting these values in the expression for the sum of the inverse n^{th} powers of the roots, we find for the sum required, the expression

$$f\,(a)+hf'(a).\phi(a)+\frac{h^2}{2}\left\{f'(a).(\phi a)^2\right\}'+\frac{h^3}{2.3}.\left\{f'\,(a).(\phi a)^3\right\}''+\&c.$$

when all but negative powers of a are excluded.

Corollary. Let $S_n=\frac{1}{\alpha^n}+\frac{1}{\beta^n}+\frac{1}{\gamma^n}+$, &c.

$$S_{n+m}=\frac{1}{\alpha^{n+m}}+\frac{1}{\beta^{n+m}}+\frac{1}{\gamma^{n+m}}\ \&c.$$

then if α be the least of the real quantities α, β, γ, &c., $\alpha^n S_n$, and $\alpha^{n+m} S_{n+m}$ converge to unity as n increases indefinitely, and therefore in the same circumstances $\frac{S_n}{S_{n+m}}$ converges towards α^m.

Putting now $\frac{1}{a^n} = f(a)$ $\frac{1}{a^{n+m}} = F(a)$, we have

$$S_n = f(a) + hf'(a)\,\phi(a) + \frac{h^2}{2}\{f'(a)\,(\phi a)^2\}' + \&c.$$

$$S_{n+m} = F(a) + h\,F'(a).\phi(a) + \frac{h^2}{2}\{F'(a)\,.(\phi a)^2\}' + \&c.$$

both series being continued to infinity when n is infinite, and we have seen that the quotient of one such series divided by another must produce a third of the same form, hence if $\frac{f(a)}{F(a)} = \psi(a) = a^m$, we have

$$a^m = \frac{S_n}{S_{n+m}} = \phi(a) + h\,\psi'(a).\phi(a) + \frac{h^2}{2}(\psi'(a)\,(\phi a)^2)'\ \&c., \textit{ ad inf.}$$

This accords with the result found immediately by the logarithmic rule for finding the m^{th} power of the least root, and demonstrates the property of such a series giving the least, when it gives a real root.

Example. To find the sum of the inverse n^{th} powers of the roots of the equation $x^2 - sx + p = 0$.

Let $\frac{p}{s} = a$; then $x = a + h\phi(x)$; $h = \frac{1}{s}$, $\phi(x) = x^2$: and $f(a) = a^{-n}$, hence $f'(a)\,\phi(a) = -na^{1-n} f'(a)$; $(\phi a)^2 = -na^{3-n}$&c., whence

$$S_n = f(a) + h\,f'(a)\phi(a) + \frac{h^2}{2}\{f'(a)(\phi a)^2\}' + \frac{h^3}{3}\{f'(a).(\phi a)^3\}''\&c.$$

$$= a^{-n} - n\,h\,a^{1-n} + \frac{n\,(n-3)}{2}.h^2 a^{2-n} - \frac{n\,(n-4)\,(n-5)}{2.3}.h^3 a^{3-n} + \&c.$$

to be terminated when the index of a is reduced to -1.

Replace now the values of a, h given by the equation, therefore

$$S_n = \frac{s^n}{p^n} - n\,\frac{s^{n-2}}{p^{n-1}} + \frac{n\,(n-3)}{2}\,\frac{s^{n-4}}{p^{n-2}} - \&c.$$

the same result which would be obtained by taking the coefficient of x^n in Log. $\left(1 - \frac{x}{a} + \frac{x^2}{h}\right)$, and then multiplying by $-n$.

(67.) Many instructive theorems may be obtained by applying the logarithmic method of solving equations. These will easily suggest themselves to the reader who has made himself familiar with the method. We give one example.

To find a series arranged according to the powers of h, of which the logarithm is the same series multiplied by h.

Let the sum of the series sought be represented by x, the conditions of the question require that Log. $(x) = hx$.

Now in this, and many other examples, a previous transformation greatly facilitates the application of the rule for finding the root. In this instance make each member of the equation the index of an exponential, of which the base is ε; that is, that of the Naperian system of logarithms, the transformed equation is $x = \varepsilon^{hx}$, the root of which is the coefficient of

$\frac{}{x}$ in $-$ Log. $\left(1-\frac{\varepsilon^{hx}}{x}\right)$; that is, in $\frac{\varepsilon^{hx}}{x}+\frac{1}{2}\cdot\frac{\varepsilon^{2hx}}{x^2}+\frac{1}{3}\cdot\frac{\varepsilon^{3hx}}{x^3}$ &c.;

and if we expand each exponential in this series, the coefficient of $\frac{1}{x}$ is readily found, viz., $1+\frac{2h}{2}+\frac{3^2h^2}{1.2.3}+\frac{4^3h^3}{1.2.3.4}+$ &c.

Now, the property imposed on this series by the conditions of the question is,

$$\text{Log.}\left\{1+h+\frac{3h^2}{1.2}+\frac{4^2h^3}{1.2.3}+\&\text{c.}\right\}=h\left\{1+h+\frac{3h^2}{1.2}+\frac{4^2h^3}{1.2.3}\&\text{c.}\right\}.$$

Suppose $h+k$ to be put for h, then x, which is a function of h, is changed to $x+x'h+x''\frac{h^2}{1.2}+$ &c., where x' x'' &c. represent the derived functions of x relative to h; thus

$$x'=1+3h+\frac{4^2h^2}{1.2}+\frac{5^3h^3}{1.2.3}\ \&\text{c.};$$

the above identity being general will remain when $h+k$ is put for h; that is,

$$\text{Log.}\left\{x+x'k+x''\frac{k^2}{1.2}+\&\text{c.}\right\}=(h+k)\left\{x+x'k+x''\frac{k^2}{1.2}\&\text{c.}\right\}$$

from which subtract the original identity, member by member, viz.,

$$\text{Log.}\,x=hx,$$

hence,

$$\text{Log.}\left\{1+\frac{x'}{x}.k+\frac{x''}{2x}.k^2\ \&\text{c.}\right\}=(x+hx')\,k+\left(x'+\frac{hx''}{2}\right)k^2+\&\text{c.},$$

and if we equate the coefficients of like powers of k, we shall obtain as many separate indentities, the first of which is,

$$\frac{x'}{x}=x+hx'$$

$$=\left(1+h+\frac{3h^2}{1.2}+\frac{4^2h^3}{1.2.3}\&\text{c}\right)+\left(h+\frac{3h^2}{1}+\frac{4^2h^3}{1.2}+\frac{5^3h^4}{1.2.3}\right)\&\text{c.}$$

$$=1+2h+\frac{3^2h^2}{1.2}+\frac{4^3h^3}{1.2.3}+\&\text{c.},$$

therefore $\left(1+3h+\frac{4^2h^2}{1.2}+\frac{5^3h^3}{1.2.3}\&\text{c.}\right)$

$$=\left(1+h+\frac{3h^2}{1.2}+\frac{4^2h^3}{1.2.3}\&\text{c.}\right).\left(1+2h+\frac{3^2h^2}{1.2}+\&\text{c.}\right)$$

Equate now the coefficient of $\frac{h^n}{1.2.3....n}$ in both members of this identity, hence

$$(n+2)^n=(n+1)^n+n.n^{n-1}+\frac{n.n-1}{1.2}.3.(n-1)^{n-2}$$
$$+\frac{n(n-1)(n-2)}{1.2.3}4^2(n-2)^{n-3}\ \&\text{c}.$$

or which, if we please, may be written in the following form:

$$n^n+\frac{3n}{2}.(n-1)^{n-1}+\frac{4^2.n(n-1)}{2\,.\,3}.(n-2)^{n-2}+\&c.=(n+2)^n-(n+1)^n;$$

provided n is a positive and integer number.

(68.) Reversion of series.

When the number of the terms in the left member of an equation is infinite, and they are arranged according to the ascending powers, the equation is of the form

$$ax + bx^2 + cx^3 + ex^4 + \&c. = \xi$$

to find the value of x, arranged in a series according to the ascending powers of ξ, is, in other words, to revert this series.

This general problem may be solved by assuming a series with indeterminate coefficients, as $x = \mathrm{A}\,.\,\xi + \mathrm{B}\,\xi^2 + \mathrm{C}\,\xi^3 + \&c.$, then

$$x^2 = \mathrm{A}^2\,\xi^2 + 2\mathrm{AB}\,\xi^3 + (\mathrm{B}^2 + 2\mathrm{AC})\,\xi^4 + \&c.$$
$$x^3 = \mathrm{A}^3\,\xi^3 + 3\mathrm{A}^2\mathrm{B}\,\xi^4 + \&c.$$
$$x^4 = \mathrm{A}^4\,\xi^4 + \&c.$$

Substitute these values in the given equation, and then compare the coefficients of like powers of ξ, you will thus find

$$a\mathrm{A} = 1$$
$$a\mathrm{B} + b\mathrm{A}^2 = 0$$
$$a\mathrm{C} + 2b\mathrm{AB} + c\mathrm{A}^3 = 0$$
$$a\mathrm{D} + b\,(\mathrm{B}^2 + 2\mathrm{AC}) + 3c\mathrm{A}^2\mathrm{B} + e\mathrm{A}^4 = 0$$
&c.,

whence $\mathrm{A} = \frac{1}{a}$

$$\mathrm{B} = -\frac{b}{a^3}$$
$$\mathrm{C} = -\frac{c}{a^4} + \frac{2b^2}{a^5}$$
$$\mathrm{D} = -\frac{e}{a^5} + \frac{5bc}{a^6} + \frac{5b^3}{a^7}$$
&c.

Example 1. Let $x + x^2 + x^3 + x^4 + \&c. = \xi$,

then $a = 1,\ b = 1,\ c = 1,\ e = 1,\ \&c.,$

therefore $\mathrm{A} = 1,\ \mathrm{B} = -1,\ \mathrm{C} = 1,\ \mathrm{D} = -1\ \&c.,$

or $x = \xi - \xi^2 + \xi^3 - \xi^4\ \&c.,$

which may be verified by observing that the given equation being the same as $\frac{x}{1-x} = \xi$, or $x = \xi - x\xi$,

we must have $x = \frac{\xi}{1+\xi} = \xi - \xi^2 + \xi^3 - \xi^4 + \&c.$

Example 2. Let $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \&c. = \xi$

$$a = 1,\quad b = \frac{1}{2},\quad c = \frac{1}{3},\quad e = \frac{1}{4},\ \&c.$$

therefore

$$\mathrm{A} = 1,\ \mathrm{B} = -\frac{1}{2},\ \mathrm{C} = \frac{1}{2} - \frac{1}{3} = \frac{1}{2.3},\ \mathrm{D} = -\frac{1}{4} + \frac{5}{6} - \frac{5}{8} = -\frac{1}{2.3.4}$$

therefore $\quad x = \xi - \frac{\xi^2}{2} + \frac{\xi^3}{2.3} - \frac{\xi^4}{2.3.4} + \text{\&c.},$

which is also obvious, by considering that, since log. $(1-x) = -\xi$, therefore $\quad x = 1 - \varepsilon^{-\xi}$.

(69.) Definition. If when $\phi(x) = y$, we have $x = f(y)$, the function represented by f is said to be inverse to that which is represented by ϕ, and the notation ϕ^{-1} is used to express this inverse function.

Problem. To find the function which is inverse to any given rational function of x, as $\phi(x)$.

If we form the equation $\phi(h) = x$, we have $h = \phi^{-1}(x)$, it is therefore only necessary to determine h in this equation.

We may put this equation under the following form:

$$h = \{x - \phi(0)\} \cdot \frac{h}{\phi(h) - \phi(0)}$$

Let $x - \phi(0) = \xi$; $\dfrac{x}{\phi(x) - (\phi 0)} = f(x)$

then $h = \xi \cdot f(h)$

from whence $h = \xi \cdot f(h) + \dfrac{\xi^2}{1.2}\{(fh)^2\}' + \dfrac{\xi^3}{1.2.3}\{(fh)^3\}'' +$, &c.

h being supposed $= 0$ in the right-hand member of this equation. therefore,

$$\phi^{-1}(x) = \{x - \phi(0)\}\left[\frac{x}{\phi(x) - \phi(0)}\right] + \frac{\{x - \phi(0)\}^2}{1.2}\left[\left(\frac{x}{\phi(x) - \phi(0)}\right)^2\right]' + \frac{\{x - \phi(0)\}^3}{1.2.3}\left[\left(\frac{x}{\phi(x) - \phi(0)}\right)^3\right]'' + \text{\&c.}$$

x being put $= 0$ in the quantities included by square brackets.

When $\phi(x)$ vanishes with x, the form of the inverse function is

$$\phi^{-1}(x) = x\left[\frac{x}{\phi x}\right] + \frac{x^2}{1.2} \cdot \left[\left(\frac{x}{\phi . x}\right)^2\right]' + \frac{x^3}{1.2.3}\left[\left(\frac{x}{\phi x}\right)^3\right]'' \text{ \&c.}$$

Example. Let $\phi(x) = ax + bx^2 + cx^3 + cx^4$, &c.

then $\dfrac{x}{\phi(x)} = (a + bx + cx^2, \text{\&c.})^{-1}$ of which the absolute term $= a^{-1}$

$\left(\dfrac{x}{\phi x}\right)^2 = (a + bx + cx^2, \text{\&c.})^{-2}$; coefficient of $x = -2ba^{-3}$

$\left(\dfrac{x}{\phi x}\right)^3 = (a + bx + cx^2, \text{\&c.})^{-3}$, when the coefficient of

$x^2 = \dfrac{1}{1.2}\left[\left(\dfrac{x}{\phi x}\right)^3\right]'' = -3ca^{-4} + 6b^2a^{-5}$

&c.

therefore $\phi^{-1}(x) = \dfrac{x}{a} - \dfrac{bx^2}{a^3} + \dfrac{(2b^2 - ac)x^3}{a^5}$, &c.

the coefficient of x^n in $\phi^{-1}(x)$, being the same as that of x^{n-1} in $\frac{1}{n}(a+bx+cx^2, \text{ \&c.})^{-n}$

The method of indeterminate coefficients may be used when there are two series, one arranged according to the powers of x, and the other to those of y; the one vanishing when $x=0$, and the other when $y=0$; thus if

$$ay+by^2+cy^3+ey^4, \text{ \&c.} = a'x+b'x^2+c'x^3+e'x^4+, \text{ \&c.}$$
$$\text{put } y = Ax+Bx^2+Cx^3+Dx^4+, \text{ \&c.}$$

then substituting and equating the coefficient of like powers of x, we have

$$aA = a' \qquad \text{therefore } A = +\frac{a'}{a}$$

$$aB + bA^2 = b' \ldots\ldots\ldots\ldots B = +\frac{b'}{a} - \frac{ba'^2}{a^3}$$

$$aC + 2bAB + cA^3 = c' \qquad C = \frac{c'}{a} - \frac{ca'^3}{a^4} - \frac{2ba'}{a^3}(b'a^2 - ba'^2)$$

&c. &c.

where, if we accent the unaccented letters, and remove the accent from the others, we shall obtain the coefficients in the series arranged according to the powers of y, by which x is expressed.

Put $\phi(x) = a'x + b'x^2 + c'x^3 + e'x^4$, &c.
$\psi(y) = ay + by^2 + cy^3 + ey^4$, &c.
then since $\psi(y) = \phi(x)$ therefore $y = \psi^{-1}\phi(x)$

But

$$\psi^{-1}\phi(x) = \phi(x)\left[\frac{x}{\psi x}\right] + \frac{(\phi x)^2}{1.2}\left[\left(\frac{x}{\psi x}\right)^2\right]' + \frac{(\phi x)}{1.2.3}\left[\left(\frac{x}{\psi x}\right)^3\right]'' + \text{ \&c.}$$

(x being ultimately put equal to zero within the square brackets), in which, if we put for $\phi(x)$ $(\phi(x))^2$, &c., their values arranged according to the powers of x, we shall obtain the same value of y as before; but in many of the astronomical applications $\phi(x)$, $(\phi x)^2$, &c., must be expressed, not in powers, but other functions, such as the circular, in which case the terms containing like multiple arcs must be taken together.

(70.) *Recurring Series* have been much used by Bernouilli, Euler, &c., in the solution of algebraical equations; we shall here trace their principal properties and applications.

Definition. Let $S = u_1 + u_2 + u_3 + u_4 + \ldots\ldots + u_x +$, &c., represent a series, of which the general, or xth term is u_x; if this term be such, that any term is the sum of a given number of preceding terms multiplied respectively by given constants, it is said to be recurring.

If m be the number of such constants, it is evident that m terms must be given, in order to form the series; these with the m constants form $2m$ arbitrary quantities necessary for the formation of the series.

The m constant multipliers are called the constants of relation: by the term *constants* is meant, that they do not vary with the number of the term of the series which they multiply.

Example 1. One constant of relation 3, and one given term of the series 2.

Series....2, 6, 18, 54, 162, &c.

thus the geometrical is the most simple of recurring series.

Example 2. Two constants of relation 1, 3, and the two first terms 2, 4.

Series.... 2, 4, 14, 46, 152, &c.

for $14=2\times1+4\times3$; $46=4\times1+14\times3$; $152=14\times1+46\times3$, &c.

Example 3. Three constants of relation, $-1, 0, 1$, and the three first terms, 1, 2, 3.

Series.. 1, 2, 3, 2, 0, -3, -5, -5, -2, 3, 8, 10, 7, -1, -11, &c.,

in which, if we subtract from any term that preceding it by two places, we get that which succeeds to it.

The sum of two recurring series, each having but one constant of relation (that is geometrical series) will be another recurring series, but with two constants of relation.

Let u_x be the general term of one geometrical series, the constant of relation, or common ratio of which is a.

Let v_x be the general term of another series of the same kind, and of which the constant of relation is β.

Let $w_x = u_x + v_x$ be the general term of the sum of the two series, when added term by term.

It remains to eliminate u_x, v_x between the three equations.

$$u_{x+1} = \alpha u_x;\ v_{x+1} = \beta v_x;\ w_x = u_x + v_x.$$

This elimination can be effected in the following manner:

$$(1)\quad w_x \ = u_x + v_x$$

therefore
$$(2)\quad w_{x+1} = u_{x+1} + v_{x+1} = \alpha u_x + \beta v_x$$

$$(3)\quad w_{x+2} = u_{x+2} + v_{x+2} = \alpha u_{x+1} + \beta v_{x+1} = \alpha^2 u_x + \beta^2 v_x.$$

Multiply (1) by an indeterminate constant λ', (2) by λ'' and equate this sum with (3), hence $w_{x+2} = \lambda' w_x + \lambda'' w_{x+1}$, under the following conditions, arising from the sums of the right-hand members of these equations.

$$\alpha^2 = \lambda' + \lambda''\alpha$$
$$\beta^2 = \lambda' + \lambda''\beta$$

therefore α, β, are the two roots of the equation $z^2 = \lambda' + \lambda'' z$, and by the theory of equations, it follows that $\lambda' = -\alpha\beta$, $\lambda'' = \alpha + \beta$, which values, it will be easily seen, satisfy these two equations;

$$\text{therefore } w_{x+2} = -\alpha\beta w_x + (\alpha + \beta) w_{x+1}$$

that is, the sum is a recurring series, of which the constants of relation are $-\alpha\beta$, $\alpha + \beta$.

The two first terms of this recurring series are given by the equations

$$w_1 = u_1 + v_1$$
$$w_2 = \alpha u_1 + \beta v_1$$

Example. Take the two geometrical series, of which the respective first terms are [illegible], and the constants of relation [illegible], viz.

First series [illegible] &c.

Second series [illegible] &c.

Then the third series, formed by adding these term by term, viz.

[illegible] &c.

is a recurring series, and since [illegible] the constants of relation in this series are [illegible]

Thus [illegible] &c.

In the very same way it appears that if three recurring series, each with one constant of relation, a for the first, b for the second, c for the third, be added [illegible] terms, the sum will be a recurring series, having three constants of relation λ, λ', λ'', that is,

$$v_{x+3} = \lambda v_{x+2} + \lambda' v_{x+1} + \lambda'' v_x$$

where $\lambda = a + b + c$, $\lambda' = -(ab + ac + bc)$, $\lambda'' = abc$.

And generally let a, a', a'', &c., be the respective constants of relation of n geometrical series, in which the general terms are u_x, u'_x, u''_x, &c., and v_x that of the series arising by taking their sum, then

$$v_x = u_x + u'_x + u''_x + \&c. \quad \ldots\ldots\ldots\ldots (1)$$

$$v_{x+1} = u_{x+1} + u'_{x+1} + u''_{x+1} + \&c. = a u_x + a' u'_x + a'' u''_x + \&c. \quad \ldots (2)$$

$$v_{x+2} = u_{x+2} + u'_{x+2} + u''_{x+2} + \&c. = a^2 u_x + a'^2 u'_x + a''^2 u''_x + \&c. \quad \ldots (3)$$

. . . .

. . . .

$$v_{x+n} = u_{x+n} + u'_{x+n} + u''_{x+n} + \&c. = a^n u_x + a'^n u'_x + a''^n u''_x + \&c. \quad \ldots (n+1)$$

Multiply the first by $\lambda^{(n)}$, 2nd by $\lambda^{(n-1)}$, ... nth by λ', and equate the sum with v_{x+n}, hence

$$v_{x+n} = \lambda^{(n)} v_x + \lambda^{(n-1)} v_{x+1} + \lambda^{(n-2)} v_{x+2} + \ldots + \lambda' v_{x+n-1}$$

provided the (n) constants λ', λ'', λ''', &c., satisfy the n equations of condition,

$$a^n = \lambda^{(n)} + \lambda^{(n-1)} a + \lambda^{(n-2)} a^2 + \ldots + \lambda' a^{n-1}$$

$$a'^n = \lambda^{(n)} + \lambda^{(n-1)} a' + \lambda^{(n-2)} a'^2 + \ldots + \lambda' a'^{n-1}$$

. . . .

. . . .

that is, a, a', a'', &c., are the n roots of the equation,

$$z^n = \lambda^{(n)} + \lambda^{(n-1)} z + \lambda^{(n-2)} z^2 + \ldots + \lambda' z^{n-1}$$

whence by the theory of equations we find

$$\lambda' = a + a' + a'' + \ldots + a^{(n-1)}$$

$$\lambda'' = -\{aa' + aa'' + a'a'' + \ldots\}$$

. . . .

$$\lambda^{(n)} = (-1)^{n-1} a a' a'' \ldots a^{(n-1)}$$

thus the constants of relation being known, and the first n terms by the equations

$$w_1 = u'_1 + u''_1 + u'''_1 \dots$$
$$w_2 = \alpha' u'_1 + \alpha'' u''_1 + \alpha''' u'''_1 + \dots$$
$$\dots\dots\dots\dots$$
$$w_n = \alpha'^{n-1} u'_1 + \alpha''^{n-1} u''_1 + \alpha'''^{(n-1)} u'''_1 + \dots$$

the recurring series for the sum is completely known.

If it should happen that the constant of relation in several (m) of the geometrical progressions was the same, then the number of the constants of relation in the recurring series would only be $n-m+1$; thus, if $\alpha' = \alpha''$ we have $u'_{x+1} = \alpha' u'_x$, $u''_{x+1} = \alpha' u''_x$; therefore if $u_x = u'_x + u''_x$, we must have $u_{x+1} = \alpha' u'_x + \alpha' u''_x = \alpha' u_x$, so that the sum of m such geometrical progressions is equivalent to one geometrical progression having a constant of relation differing from those of the $n-m$ remaining series.

If the terms of a recurring series $u_1 + u_2 + u_3 +$ &c. be multiplied by the corresponding terms of the geometrical series, $1 + z + z^2 +$ &c., term by term, the series resulting will be another recurring series.

For let $\lambda', \lambda'', \lambda''' \dots \lambda'''^{(n)}$ be the constants of relation in the given recurring series, or let

$$u_{x+n} = \lambda' u_x + \lambda'' u_{x+1} + \lambda''' u_{x+2} + \dots + \lambda'''^{(n)} u_{x+n-1}$$

therefore $u_{x+n} z^{x+n-1} = \lambda' z^n . u_x z^{x-1} + \lambda'' z^{n-1} . u_{x+1} z^x + \lambda'' z^{n-2} . u_{x+2} z^{x+1} + \dots$
$\lambda'''^{n} z . u_{x+n-1} . z^{x+n-2}$.

Now the product of the two series, taken term by term, is the series

$$u_1 + u_2 z + u_3 z^2 + u_4 z^3, \text{ \&c.}$$

which, by the preceding equation, must be a recurring series, of which the constants of relation are $\lambda' z^n$, $\lambda'' z^{n-1}$, $\lambda''' z^{n-2} \dots \lambda'''^{(n)} z$.

(71.) A recurring series, such as this, which is arranged according to the powers of some quantity z, is merely the expansion of a rational fraction, of which the denominator is of n, and the numerator of inferior dimensions.

Represent the expansion of the rational fraction

$$\frac{a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots + a_{n-1} z^{n-1}}{1 - \lambda' z^n - \lambda'' z^{(n-1)} - \lambda''' z^{(n-2)} - \dots - \lambda'''^{(n)} z}$$

by the series

$$u_1 + u_2 z + u_3 z^2 + \dots + u_n z^{n-1} + u_{n+1} z^n + \dots + u_{x+n} z^{x+n-1} + \text{, \&c.}$$

If we multiply this series by the denominator of the fraction, and arrange the product according to the powers of z, it is clear that the coefficients of higher powers of z than the $(n-1)$th, must vanish.

Now the coefficient of z^{x+n-1} in this product is visibly

$$u_{x+n} - \lambda'''^{(n)} u_{x+n-1} - \lambda'''^{(n-1)} u_{x+n-2} - \dots - \lambda'' u_{x+1} - \lambda' u_x$$

therefore $u_{x+n} = \lambda' u_x + \lambda'' u_{x+1} + \lambda''' u_{x+2} + \dots \quad \lambda'''^{(n)} u_{x+n-1}$

consequently the coefficients u_x form a recurring series, of which the constants of relation are $\lambda', \lambda'', \lambda''' \dots \lambda'''^{(n)}$; but in order that the fraction may be completely identical with the series, the first n terms

must have the following relations with the numerator of the fraction, viz.,

$$a_0 = u_1$$
$$a_1 = u_2 - \lambda^{''''(n)} u_1$$
$$a_2 = u_3 - \lambda^{'''(n-1)} u_1 - \lambda^{''(n)} u_2$$
$$a_3 = u_4 - \lambda^{'''(n-2)} u_1 - \lambda^{'''(n-1)} u_2 - \lambda^{''(n)} u_3$$
$$\cdots\cdots\cdots\cdots$$
$$a_{n-1} = u_n - \lambda' u_1 - \lambda'' u_2 - \ldots - \lambda^{'''(n)} u_{n-1}$$

Conversely, by these equations, when the first n terms of a recurring series and the constants of relation are given, we can find the rational fraction by the expansion of which it is produced.

Example 1. To find the rational fraction which generates a recurring series, of which the two first terms are $2+4z$, and the constants of relation are z^2 and $3z$, viz.,

$$2+4z+14z^2+46z^3+152z^4+ \ \&c.$$
$$\lambda'=1 \quad \lambda''=3 \quad u_1=2 \quad u_2=4$$
$$a_0=2 \quad a_1=4-6=-2$$

the fraction required is therefore $\dfrac{2-2z}{1-z^2-3z}$.

Verification $1-3z-z^2)2-2z \quad (2+4z+14z^2+46z^3+ \ \&c.$

$$\begin{array}{l} 2-6z-2z^2 \\ \hline 4z+2z^2 \\ 4z-12z^2-4z^3 \\ \hline 14z^2+4z^3 \\ 14z^2-42z^3-14z^4 \\ \hline 46z^3+14z^4 \\ 46z^3-138z^4-46z^5 \\ \hline \cdots\cdots \end{array}$$

Example 2. To find the rational fraction which generates the series

$$1+2z+3z^2+2z^3+z^5-3z^6-5z^7-5z^8-2z^9+3z^{10}+ \ \&c.$$

when $u_1=1 \quad u_2=2 \quad u_3=3 \quad \lambda'=-1 \quad \lambda''=0 \quad \lambda'''=1$

$$a_0=1 \quad a_1=2-1=1$$
$$a_2=3-1.2=1$$

The fraction required is therefore $\dfrac{1+z+z^2}{1+z^3-z}$.

DECOMPOSITION OF RATIONAL FRACTIONS.

(72.) We have seen that the expansion of rational fractions produces recurring series; now each such fraction can generally be decomposed into simple fractions, each of which (with exceptions in the case of equal roots) would generate geometrical series, agreeably with the theorem before proved, that the sum of any number of geometrical series, added in corresponding terms, is a recurring series.

Let α', α'', α''', ... $\alpha'''^{(n)}$ be all the roots of the equation $\phi(z)=0$, then $\phi(z)=C.(z-\alpha')\ (z-\alpha'')\ (z-\alpha''')....(z-\alpha'''^{(n)})$, the quantity C being independent of z.

Let $f(z)$ be another function of z, of dimensions inferior to n; the most general form of $f(z)$ will be $A_0+A_1z+A_2z^2+....+A_{n-1}z^{n-1}$. which contains n constants.

Suppose now we reduce to a common denominator, and add together the n simple fractions in the following expression:

$$\frac{A}{z-\alpha'}+\frac{A''}{z-\alpha''}+\frac{A'''}{z-\alpha'''}+....+\frac{A'''^{(n)}}{z-\alpha'''^{(n)}}.$$

the common denominator, or that of the sum, will be $\frac{1}{C}\phi(z)$, and the numerator will be a function of z of $n-1$, dimensions, viz.

$A'(z-\alpha'')(z-\alpha''')\ ...\ (z-\alpha'''^{(n)})+A''(z-\alpha')(z-\alpha''')...(z-\alpha'''^{(n)})+$ &c.

which, being arranged according to the powers of z, may be made identical with $f(z)$, by determining A', A'', &c., so as to satisfy the n equations,

$$\begin{array}{lr}
A' + A'' + A''' + \ldots\ldots\ldots A''^{(n)} = & A_{n-1} \\
A'(\alpha''+\alpha'''+\ldots)+A''(\alpha'+\alpha'''+,\ \&c.) \ \ldots\ldots = & -A_{n-2} \\
A'(\alpha''\alpha'''+\ldots)+A''(\alpha'\alpha'''+,\ \&c.) \ \ldots\ldots = & A_{n-3} \\
\&c. \qquad\qquad \&c. &
\end{array}$$

But it should be observed, that if two or more of the roots α', α'', &c., be equal, then we should have fewer unknown quantities than equations: thus if $\alpha'=\alpha''$, it is clear by inspection of these equations that A', A'' have the same multipliers, so that they enter in all in the form $A'+A''$, which is also obvious, since the simple fractions $\frac{A'}{z-\alpha'}+\frac{A''}{z-\alpha''}$ would then be the same as $\frac{A'+A''}{z-\alpha'}$; therefore, this is a case of exception, in which n simple fractions are incapable of producing $\frac{f(z)}{\phi(z)}$ as their sum, by simple fractions being meant, such that z is only of the first degree, or is linear in the denominator and the numerator constant: this case of equal roots we shall at present reserve, and suppose α', α'', &c., to be all unequal.

The early analysts generally sought the numerators A', A'' &c., by elimination between the preceding equations. The following method is, however, more expeditious, in which we suppose $C=1$, for if it had any other value, we could take it into the coefficients of the numerator by division.

Since

$$\frac{f(z)}{(z-\alpha')\ (z-\alpha'')....(z-\alpha'''^{(n)})}=\frac{A'}{z-\alpha'}+\frac{A''}{z-\alpha''}+....\frac{A'''^{(n)}}{z-\alpha'''^{(n)}};$$

therefore

$$\frac{f(z)}{(z-\alpha'')(z-\alpha''')\ ...\ (z-\alpha'''^{(n)})}=A'+\frac{A''(z-\alpha')}{z-\alpha''}+\frac{A'''(z-\alpha')}{z-\alpha'''}+\&c.$$

This, when A', A'', &c. have been properly determined, must be an absolute identity, whatever value we assign to z; for A', A'', &c., being

constants, that is, not containing z, can be in no way affected by putting for z any value at pleasure, as they depend only on α', α'', α''', &c.

Put therefore $z = \alpha'$, and this identity becomes

$$\frac{f(\alpha')}{(\alpha'-\alpha'')(\alpha'-\alpha''')\ldots\ldots(\alpha'-\alpha'''^{(n)})} = \mathrm{A}'.$$

We may get more simply this expression for A′, by observing that, since $\phi(z) = (z-\alpha')(z-\alpha'')\ldots\ldots(z-\alpha'''^{(n)})$; therefore
$\phi'(z) = (z-\alpha'')(z-\alpha''')\ldots\ldots(z-\alpha'''^{(n)}) + (z-\alpha')(z-\alpha''')\ldots\ldots$
$+ (z-\alpha')(z-\alpha'')(z-\alpha^{iv})\ldots\ldots + \&c.$
where $\phi'(z)$ denotes the derived function from $\phi(z)$; hence
$\phi'(\alpha') = (\alpha'-\alpha'')(\alpha'-\alpha''')\ldots\ldots\ldots(\alpha'-\alpha'''^{(n)})$

Therefore $\quad \mathrm{A}' = \dfrac{f(\alpha')}{\phi'(\alpha')}$

similarly $\quad \mathrm{A}'' = \dfrac{f(\alpha'')}{\phi'(\alpha'')}$

&c.;

that is, if we write for z the successive roots α', α'', &c., in the formula $\dfrac{f(z)}{\phi'(z)}$, we shall obtain the successive numerators of the simple fractions; and it is obvious that the same formulæ hold true if $\phi(z)$ should contain a constant multiplier C.

Example 1. To decompose the fraction $\dfrac{1}{(z-\alpha)(z-\beta)}$,

here $\quad f(z) = 1$; $\phi(z) = z^2 - (\alpha+\beta)z - \alpha\beta$, $\phi'(z) = 2z - \alpha + \beta$;

therefore $\quad \dfrac{f(z)}{\phi'(z)} = \dfrac{1}{2z-(\alpha+\beta)}$.

Put α, β successively for z, and we obtain the numerators of the simple fractions; viz., $\dfrac{1}{\alpha-\beta}$ and $\dfrac{1}{\beta-\alpha}$;

therefore $$\frac{1}{(z-\alpha)(z-\beta)} = \frac{1}{\alpha-\beta}\left(\frac{1}{z-\alpha} - \frac{1}{z-\beta}\right).$$

Example 2. To decompose $\dfrac{2z+1}{z.(z+1)(z+2)}$.

The roots of the denominator equated to zero are $0, -1, -2$, which, being substituted for z in $\dfrac{f(z)}{\phi'(z)}$, or, which is the same, in

$\dfrac{2z+1}{(z+1)(z+2)+z(z+2)+z(z+1)}$, give the numerators of the simple fractions; viz.,

$$\frac{1}{2}, +1, -\frac{3}{2}, \text{ or } \frac{2z+1}{z.(z+1)(z+2)} = \frac{1}{2}\left(\frac{1}{z} + \frac{2}{z+1} - \frac{3}{z+2}\right).$$

All the numerators are unity when $f(z)$ is itself the derived function of $\phi(z)$, for then $\dfrac{f(z)}{\phi'(z)} = 1$, whatever value be assigned to z. Hence, if α', α'', &c., be the roots of the equation $\phi(z) = 0$, we have

$$\frac{\phi'(z)}{\phi(z)}=\frac{1}{z-\alpha'}+\frac{1}{z-\alpha''}+\frac{1}{z-\alpha'''}\ \&c.$$

Example. To decompose $\frac{1}{\varepsilon^{hz}-1}$ into simple fractions.

First, we change the form of the proposed fraction thus;

$$\frac{1}{\varepsilon^{hz}-1}=\frac{\varepsilon^{-\frac{h}{2}.z}}{\varepsilon^{\frac{h}{2}z}-\varepsilon^{-\frac{h}{2}z}}=\frac{1}{2}.\frac{\varepsilon^{\frac{h}{2}z}+\varepsilon^{-\frac{h}{2}.z}}{\varepsilon^{\frac{h}{2}.z}-\varepsilon^{-\frac{h}{2}.z}}-\frac{1}{2}$$

Now, if m denote any integer positive or negative including zero, and π be the semi-circumference of a circle of which the radius is unity, the principles of trigonometry give $\varepsilon^{m\pi\sqrt{-1}}-\varepsilon^{-m\pi\sqrt{-1}}=0$. Hence, if we put $\varepsilon^{\frac{h}{2}z}-\varepsilon^{-\frac{h}{2}.z}=0$, we have $z=\frac{2m\pi\sqrt{-1}}{h}$: the values of z are therefore in number infinite, viz.

$$0,\ \frac{2\pi\sqrt{-1}}{h},\ \frac{4\pi\sqrt{-1}}{h},\ \frac{6\pi\sqrt{-1}}{h},\ \&c.$$

$$-\frac{2\pi\sqrt{-1}}{h},\ \frac{-4\pi\sqrt{-1}}{h},\ \frac{-6\pi\sqrt{-1}}{h},\ \ldots\ \&c.$$

Now make $\phi(z)=\varepsilon^{\frac{h}{2}.z}-\varepsilon^{-\frac{h}{2}.z}=2\left\{\frac{h}{2}.z+\left(\frac{h}{2}\right)^3.\frac{z^3}{1.2.3}\right.$

$$\left.+\left(\frac{h}{2}\right)^5.\frac{z^5}{1.2.3.4.5}+\&c.\right\};$$

therefore $\phi'(z)=h\left\{1+\left(\frac{h}{2}\right)^2.\frac{z^2}{1.2}+\left(\frac{h}{2}\right)^4.\frac{z^4}{1.2.3.4}+\&c.\right\}$

$$=\frac{h}{2}\left(\varepsilon^{\frac{h}{2}.z}+\varepsilon^{-\frac{h}{2}.z}\right).$$

Hence $\frac{1}{\varepsilon^{hz}-1}=\frac{1}{h}.\frac{\phi'(z)}{\phi(z)}-\frac{1}{2}$

$$=-\frac{1}{2}+\frac{1}{h}\left\{\begin{array}{l}\frac{1}{z}+\frac{1}{z-\frac{2\pi\sqrt{-1}}{h}}+\frac{1}{z-\frac{4\pi\sqrt{-1}}{h}}+\&c.\\[2ex] +\frac{1}{z+\frac{2\pi\sqrt{-1}}{h}}+\frac{1}{z+\frac{4\pi\sqrt{-1}}{h}}+\&c.\end{array}\right\}.$$

Corollary 1. Since

$$\frac{1}{z+\frac{2\pi\sqrt{-1}}{h}}=\frac{h}{2\pi\sqrt{-1}}+\frac{h^2}{2^2\pi^2}.z-\frac{h^3}{2^3\pi^3\sqrt{-1}}.z^2-\frac{h^4}{2^4\pi^4}.z^3+\&c.,$$

therefore $\frac{1}{z-\frac{2\pi\sqrt{-1}}{h}}+\frac{1}{z+\frac{2\pi\sqrt{-1}}{h}}=2h\left\{\frac{1}{2^2\pi^2}.hz-\frac{1}{2^4\pi^4}.h^3z^3+\&c.\right\}.$

Expanding similarly the other corresponding pairs of simple fractions, and collecting all the coefficients of like terms, we have

$$\frac{1}{\varepsilon^{hz}-1}=\frac{1}{hz}-\frac{1}{2}+\frac{2hz}{\pi^2}\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\&c.\right)$$
$$\frac{2h^3z^3}{\pi^4}\left(\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\&c.\right)$$
$$+\&c.$$

Let
$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\&c.=2B_1\pi^2$$
$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\&c.=\frac{2^3B_3\pi^4}{1.2.3}$$
$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\&c.=\frac{2^5B_5\pi^6}{1.2.3.4.5}$$
&c. &c.;

we have then $\frac{1}{\varepsilon^{hz}-1}=\frac{1}{hz}-\frac{1}{2}+B_1.hz+B_3.\frac{h^3z^3}{1.2.3}+B_5.\frac{h^5z^5}{1.2.3.4.5}$;

from which we see that, though $\varepsilon^{hz}-1$ consists of both even and odd powers of z, its reciprocal contains only odd powers and a constant.

The numbers B_1, B_3, B_5, &c., may be easily calculated without summing the preceding series. Thus, since (putting $h=1$)

$$\frac{\phi'(z)}{\phi(z)}=\frac{1}{\varepsilon^z-1}+\frac{1}{2}=\frac{1}{z}+B_1z+B_3\frac{z^3}{1.2.3}+\&c.$$

And again,

$$\text{Log. }\phi(z)=\text{Log. }z+\text{Log.}\left\{1+\left(\frac{1}{2^2}.\frac{z^2}{2.3}+\frac{1}{2^4}.\frac{z^4}{2.3.4.5}+\&c.\right)\right\}.$$

Suppose the latter logarithm, expanded and arranged according to the powers of z^2, to be $A_1z^2-A_3z^4+A_5z^6-$&c., then, taking the derived functions relative to z, we have also

$$\frac{\phi'(z)}{\phi(z)}=\frac{1}{z}+2A_1z-4A_3z^3+6A_5z^5-\&c.;$$

therefore $B_1=1.2A_1$, $B_3=1.2.3.4A_3$, $B_5=1.2.3.4.5.6A_5$, &c.

But $A_1=\frac{1}{2^2.2.3}$, $A_3=\frac{1}{2^5.2^2.3^2}-\frac{1}{2^5.3.4.5}=\frac{1}{2^4.3^2.4.5}$, &c.;

therefore $B_1=\frac{1}{2^2.3}$, $B_3=\frac{1}{2^3.3.5}$, &c.

Corol'ary 2. From hence the sums of the even negative powers of the natural numbers are easily known.

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots \textit{ad inf.}=2B_1\pi^2=\frac{\pi^2}{6}$$
$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ \ldots\ldots\ldots=\frac{2^3B_3\pi^4}{1.2.3}=\frac{\pi^4}{90}$$
&c. &c.

The expansion of $\frac{1}{\varepsilon^{hz}-1}$ according to the powers of z above

found, admits of a remarkable extension, which has great use in the summation of series, and it may be easily deduced therefrom, as follows:—

First, we have

$$hz-(\varepsilon^{hz}-1)+\tfrac{1}{2}hz(\varepsilon^{hz}-1)-B_1h^2z^2(\varepsilon^{hz}-1)+B_3\frac{h^4z^4}{1.2.3}(\varepsilon^{hz}-1)\&c.=0.$$

With the left member of this equation compare the following series:—

$$hf'(x)-\big(f(x+h)-f(x)\big)+\tfrac{1}{2}h\big(f(x+h)-f(x)\big)'$$
$$-B_1h^2\{f(x+h)-f(x)\}''+\&c.$$

It will be seen immediately (by putting for ε^{hz}, in the former, its expansion $1+hz+\frac{h^2z^2}{1.2}$, &c., and for $f(x+h)$, in the latter, its value $f(x)+f'(x).h+f''(x).\frac{h^2}{1.2}+\&c.$) that the coefficient of z in *any term* of the first series is the same as the coefficient of $f'(x)$ in the corresponding term of the latter; that the coefficient of z^2 is the same as that of $f''(x)$, &c.; and, since the first series is identically zero, the coefficients of each power of z collected from the different terms must separately be equal to zero; and therefore the same is true of the coefficients of $f'(x)$, $f''(x)$, &c., in the second series: consequently this second and much more general series must also be identically nothing.

$$\text{Let } F(x)=f(x)-\tfrac{1}{2}hf'(x)+B_1h^2f''(x)-\frac{B_3h^4}{1.2.3}.f^{\text{iv}}(x)$$
$$+\frac{B_5h^6}{1.2.3.4.5}.f^{\text{vi}}(x)-\&c.;$$

then the preceding general theorem is equivalent to this:

$$F(x+h)-F(x)=hf'(x).$$

Example 1. Let $f(x)=x$, and consequently $F(x)=x-\frac{h}{2}$; therefore $F(x+h)=x+\frac{h}{2}$, and $F(x+h)-F(x)=h$, which agrees with the theorem, since $f'(x)$ is in this case equal to unity.

Example 2. Let $f(x)=x^2$, and therefore $F(x)=x^2-hx+\frac{h^2}{6}$, from whence $F(x+h)-F(x)=(2x+h)h-h^2=2xh=hf'(x)$.

73. The application of this theorem to the summation of series is the converse of that contained in the preceding examples; that is, $F(x+1)-F(x)$ is given, $f'(x)$ which is equal to this difference, is therefore known; hence the successive derived functions $f''(x)$, $f'''(x)$, &c., are easily found; and if we can find $f(x)$, the function from which $f'(x)$ is derived, then $F(x)$ may be found by the theorem $F(x)=f(x)-\frac{1}{2}f'(x)+B_1f''(x)-\frac{B_3}{1.2.3}f^{\text{iv}}(x)$, &c.

Let v_1, v_2, v_3, &c., be the terms of a series which it is proposed to sum, and let $F(x)$ represent the sum of x terms; then

$$F(x)=v_1+v_2+v_3+\dots.+v_x;$$

therefore $F(x+1)=u_1+u_2+u_3+\ldots.+u_x+u_{x+1}$;

whence $F(x+1)-F(x)=u_{x+1}$.

We must consider u_{x+1} as $f'(x)$, and then find $F(x)$, as above described. It is only necessary to remark that, since we are to reascend to $f(x)$ from $f'(x)$, it will be necessary to add an arbitrary constant to any particular value found for $f(x)$, since, if C be a quantity independent of x, then $(f(x)+C)'=f'(x)$. This constant merely determines where the series is supposed to commence: if the sum, for instance, be taken from the first term, then $F(1)=u_1$, which equation will determine the constant.

Having given this general theorem as following from the theory of rational fractions, we shall only give one example of its application, since our object in this treatise is not to discuss the summation of series except when the latter subject is correlative to the theory of equations.

To find the sum of x terms of the series $1^3+2^3+3^3+$ &c.

$$\text{Let } F(x)=1^3+2^3+3^3+4^3+\ldots.+x^3$$

$$F(x+1)=1^3+2^3+3^3+4^3+\ldots.+x^3+(x+1)^3;$$

therefore $f'(x)=F(x+1)-F(x)=(x+1)^3$

$$f'''(x)=3(x+1)^2$$

$$f^{\text{iv}}(x)=6,$$

which, being constant, needs not to be written separately, but may be comprised in the arbitrary C of the function, of which $(x+1)^3$ is the derived, viz. $\frac{1}{4}(x+1)^4+C$.

Hence $F(x)=C+\frac{1}{4}(x+1)^4-\frac{1}{2}(x+1)^3+\frac{1}{4}(x+1)^2$.

To determine C, put $x=1$, observing that $F(1)$ is merely the first term; therefore $1=C+\frac{1}{4}(2^4-2^4+2^2)$, or $C=0$;

$$\text{therefore } F(x)=(x+1)^2\{(x+1)^2-2(x+1)+1\}$$

$$=\left\{\frac{x.(x+1)}{2}\right\}^2;$$

that is, $1^3+2^3+3^3+\ldots.+x^3=\{1+2+3+\ldots.+x\}^2$.

From this the learner will see how to apply the theorem to other examples, the only difficulty being to find $f(x)$ from $f'(x)$, which is given; and therefore its more extensive applications must be postponed until he has acquired a knowledge of the integral calculus.

74. We are now to consider the decomposition of fractions when the denominator, equated with zero, contains several equal roots, as in the fraction $\dfrac{f(z)}{(z-\alpha')^m(z-\alpha'')(z-\alpha''')\ldots(z-\alpha'''^{(n)})}$, in which the dimensions of the numerator are supposed inferior to those of the denominator.

Put, as before, $\phi(z)=(z-\alpha')(z-\alpha'')(z-\alpha''')\ldots.(z-\alpha'''^{(n)})$,

then $\dfrac{1}{\phi(z)}=\dfrac{1}{\phi'(\alpha')}\cdot\dfrac{1}{z-\alpha'}+\dfrac{1}{\phi'(\alpha'')}\cdot\dfrac{1}{z-\alpha''}+\dfrac{1}{\phi'(\alpha''')}\cdot\dfrac{1}{(z-\alpha''')}+$ &c.;

and since $f(z)=\{f(z)-f(\alpha')\}+f(\alpha')=\{f(z)-f(\alpha'')\}+f(\alpha'')=$ &c.;

therefore $\dfrac{f(z)}{\phi(z)}=\dfrac{1}{\phi'(\alpha')}\cdot\dfrac{f(z)-f(\alpha')}{z-\alpha'}+\dfrac{1}{\phi(\alpha'')}\cdot\dfrac{f(z)-f(\alpha'')}{z-\alpha''}+$ &c.

$$+\frac{1}{\phi'(\alpha')}\cdot\frac{f(\alpha')}{z-\alpha'}+\frac{1}{\phi'(\alpha'')}\cdot\frac{f(\alpha'')}{z-\alpha''}+\text{&c.},$$

the uppermost line obviously being an integer function of z, and the lower purely fractional.

Suppose that $\alpha'+h$ is written for α' in each term of this identity, and that the terms thus arising are expanded according to the ascending powers of h, and finally that the coefficients of h^{m-1} on both sides of the sign $=$ are equated, this will give the required decomposition.

$\phi(z)$ is changed into $(z-\alpha'-h)(z-\alpha'')(z-\alpha''')\ldots(z-\alpha''''^{(n)})$; or, which is the same, it becomes $\dfrac{\phi(z)}{z-\alpha'}.(z-\alpha'-h)=\phi(z)\left(1-\dfrac{h}{z-\alpha'}\right)$; therefore $\dfrac{f(z)}{\phi(z)}$ becomes $\dfrac{f(z)}{\phi(z)}.\left(1-\dfrac{h}{z-\alpha'}\right)^{-1}$; and if we expand this negative power we find the coefficient of $h^{m-1}=\dfrac{f(z)}{(z-\alpha')^{m-1}\phi(z)}$; that is, the proposed proper fraction itself.

The uppermost line of the identity, as has been observed, is an integer function of z, and will be such when $\alpha'+h$ is put for α'; and therefore the coefficient of h^{m-1}, taken throughout the whole of this line, must be identically zero, because otherwise we should have an integer function of z equal to another function of the same which was a proper fraction, and that is impossible.

We have therefore only to consider the coefficients of h^{m-1}, when $\alpha'+h$ is put for α' in the lower line.

The first term is altered to $\dfrac{f(\alpha'+h)}{\phi'(\alpha'+h)}.\dfrac{1}{z-\alpha'-h}$, and, expanding each factor according to the powers of h, we may write it thus:

$$\left\{\frac{f(\alpha')}{\phi'(\alpha')}+h\left(\frac{f\alpha'}{\phi'\alpha'}\right)'+\frac{h^2}{1.2}.\left(\frac{f\alpha'}{\phi'\alpha'}\right)''+\&c.\right\}.\left\{\frac{1}{z-\alpha'}+\frac{h}{(z-\alpha')^2}\right.$$
$$\left.+\frac{h^2}{(z-\alpha')^3}+\&c.\right\};$$

therefore the coefficient of h^{m-1} in this term is

$$\frac{1}{(z-\alpha')^m}\left\{\frac{f(\alpha')}{\phi'(\alpha')}+(z-\alpha')\left(\frac{f\alpha'}{\phi'\alpha'}\right)'+\frac{(z-\alpha)^2}{1.2}.\left(\frac{f\alpha'}{\phi'\alpha'}\right)''+\ldots\right.$$
$$\left.+\frac{(z-\alpha')^{m-1}}{1.2.3\ldots(m-1)}.\left(\frac{f\alpha'}{\phi'\alpha'}\right)^{'''(m-1)}\right\};$$

in other words, it is that part of $\dfrac{f(z)}{(z-\alpha')^m(z-\alpha''),\ \&c.}$ which contains the negative powers of $z-\alpha'$.

The coefficients of h^{m-1} in the remaining terms are much more easily found; thus, since $\phi'(\alpha'')=(\alpha''-\alpha')(\alpha''-\alpha''')\ldots(\alpha''-\alpha''''^{(n)})$, the change of α' into $\alpha'+h$ converts $\phi'(\alpha'')$ into $\phi'(\alpha'').\dfrac{\alpha''-\alpha'-h}{\alpha''-\alpha'}$ $=\phi'(\alpha'').\left(1-\dfrac{h}{\alpha''-\alpha'}\right)$, and $\dfrac{1}{\phi'(\alpha'')}$ into $\dfrac{1}{\phi'(\alpha'')}.\left(1-\dfrac{h}{\alpha''-\alpha'}\right)^{-1}$, while this change does not at all affect $\dfrac{f(\alpha'')}{z-\alpha''}$; therefore the coefficient of

h^{m-1} in the second term of the lower line, when $a'+h$ is put for a', is $\frac{1}{(a''-a')^{m-1}\phi'(a'')}\cdot\frac{f(a'')}{z-a''}$; and a similar expression is obtained for the third term, writing a''' for a'', and so on.

Collecting now all these coefficients of h^{m-1}, and equating with the given fraction, we have

$$\frac{f(z)}{(z-a')^{m-1}\phi(z)}=\frac{f(a')}{\phi'(a')}\cdot\frac{1}{(z-a')^{m}}+\left(\frac{fa'}{\phi'a'}\right)'\cdot\frac{1}{(z-a')^{m-1}}$$

$$+\left(\frac{fa'}{\phi'a'}\right)''\cdot\frac{1}{1.2(z-a)^{m-2}}, \text{ \&c.}$$

$$+\frac{f(a'')}{(a''-a')^{m-1}\phi'(a'')}\cdot\frac{1}{z-a''}+\frac{f(a''')}{(a'''-a')^{m-1}\phi'(a''')}\cdot\frac{1}{z-a'''}+\text{\&c.},$$

observing to continue the upper line only while it contains negative powers of $z-a'$, and to cease at the term which contains $(z-a')^{-1}$.

It is easy to see that $(a''-a')^{m-1}\phi'(a'')=\{(z-a')^{m-1}\phi z\}'$, when z is put equal to a''; and therefore the coefficients of $\frac{1}{z-a''}$, $\frac{1}{z-a'''}$, &c. may always be found by the application of the same rule as that for unequal roots.

Example 1. Decompose $\frac{1}{z^m(z-1)}$.

Take the fraction $\frac{1}{(z-a')^m(z-a'')}$, and finally make $a'=0$, $a''=1$. In this case $f(z)=1$, $\phi(z)=(z-a')(z-a'')$, $\phi'(z)=2z-(a'+a'')$; and therefore $\phi'(a')=a'-a''$, and $\frac{f(a')}{\phi'(a')}=\frac{1}{a'-a''}$, $\left(\frac{fa'}{\phi'a'}\right)'=-\left(\frac{1}{a'-a''}\right)^2$, $\left(\frac{f(a')}{\phi a'}\right)''=1.2\left(\frac{1}{a'-a''}\right)^3$, &c., $\frac{f(a'')}{(a''-a')^{m-1}\phi'(a'')}=\frac{1}{(a''-a')^m}$.

Put now, for a', a'' their values in the general formula.

Hence. $\frac{1}{z^m(z-1)}=-\frac{1}{z^m}-\frac{1}{z^{m-1}}-\frac{1}{z^{m-2}}\ldots-\frac{1}{z}+\frac{1}{z-1}$,

which we should also deduce if we write the proposed fraction in the form $-\frac{1-z^m}{1-z}\cdot\frac{1}{z^m}+\frac{1}{z-1}$.

Example 2. Decompose $\frac{2z^3+7z^2+6z+2}{z^4+3z^3+2z^2}$.

The roots of the denominator equated to zero are 0, 0, -2, -1. The numerators of the fractions of which the denominators are $z+2$, $z+1$, will be obtained by the rule for unequal roots by writing -2, -1, successively for z in $\frac{2z^3+7z^2+6z+2}{4z^3+9z^2+4z}$: they are $-\frac{1}{2}$ and 1.

Again, in this case $\phi(z)=(z+1)(z-a')(z+2)$ a' being finally made zero, hence $\phi'(a')=(a'+1)(a'+2)$.

therefore $\frac{f(\alpha')}{\phi'(\alpha')}=\frac{2+6\alpha'+7\alpha'^2+2\alpha'^3}{2+3\alpha'+\alpha'^2}=1+\frac{3}{2}\alpha'+\frac{3}{4}\alpha'^2$ &c.

and making $\alpha'=0$ we find $\frac{f(\alpha')}{\phi'(\alpha')}=1$ $\left(\frac{f(\alpha')}{\phi'(\alpha')}\right)'=\frac{3}{2}$; therefore the required partial fractions are $-\frac{1}{2}\cdot\frac{1}{z+2}$, $\frac{1}{z+1}$, $\frac{1}{z^2}$, $\frac{3}{2}\frac{1}{z}$.

In like manner if we had several sets of equal roots as in the fraction

$$\frac{f(z)}{(z-\alpha')^m(z-\alpha'')^p(z-\alpha''')^q\ \&c.}$$

We have only to decompose $\frac{f(z)}{(z-\alpha')(z-\alpha'')(z-\alpha'''\ \&c.}$, and then put $\alpha'+h$ for α' $\alpha''+k$ for α'' $\alpha'''+l$ for α''' &c. and select from the component fractions the coefficient of $h^{m-1}\,k^{p-1}\,l^{q-1}$.....and we shall have the required simple fractions.

75. The decomposition of rational fractions in the case of equal roots may however be more readily effected by the following theorem, which, I believe, has not been before given.

Let $\frac{P}{Q}$ be a proper rational fraction, the numerator and denominator being functions of z, and the latter containing a factor repeated n times, that is $Q=(z-\alpha')^n\,.\,Q_1$.

Expand $\frac{P}{Q_1}$ in the form $A_0+A_1(z-\alpha')+A_2(z-\alpha')^2+\ldots\ldots$

$$+A_{n-1}(z-\alpha')^{n-1}+\ \&c.$$

then $\frac{A_0}{(z-\alpha')^n}$, $\frac{A_1}{(z-\alpha')^{n-1}}$, $\frac{A_2}{(z-\alpha')^{n-2}}\ldots\ldots\frac{A_{n-1}}{z-\alpha'}$ will be partial fractions.

Again, let $(z-\alpha'')^p$ be another factor of Q, or $Q=(z-\alpha'')^p\,.\,Q_2$ then expand $\frac{P}{Q_2}$ in the form $B_0+B_1(z-\alpha'')+B_2(z-\alpha'')^2+\ldots\ldots$

$$B_{p-1}(z-\alpha'')^{p-1}+\&c.$$

then $\frac{B_0}{(z-\alpha'')^p}$, $\frac{B_1}{(z-\alpha'')^{p-1}}$, $\frac{B_2}{(z-\alpha'')^{p-2}}\ldots\ldots\frac{B_{p-1}}{z-\alpha''}$ are also partial fractions, and proceed in the same way for all the different factors of Q.

The given fraction $\frac{P}{Q}$ will be the sum of all these partial fractions.

For it is evident that all the partial fractions which have no power of $z-\alpha'$ as denominator may be expanded in the ascending powers of $(z-\alpha')$, all with positive indices, and they would give in the corresponding value of $\frac{P}{Q_1}$ no power of $z-\alpha'$ inferior to the nth; the powers which are inferior to the nth in the expansion of $\frac{P}{Q_1}$ must, therefore, arise from those terms only which contain $z-\alpha'$ in their denominator; consequently if the first n terms of $\frac{P}{Q_1}$ be $A_0+A_1(z-\alpha')+A_2(z-\alpha')^2+\ldots$

$A_{n-1}(z-a')^n$ the partial fractions of $\frac{P}{Q}$ which have $z-a'$ in the denominator must be $\frac{A_0}{(z-\alpha)^n}+\frac{A_1}{(z-\alpha)^{n-1}}$ &c., and the same reasoning applies equally to the partial fractions containing $z-a''$ &c.

Example. 1. Decompose $\left(\frac{1}{z^2-z}\right)^2 = Z$.

First: $Z = \frac{1}{z^2}\cdot(1-z)^{-2} = \frac{1}{z^2} + \frac{2}{z} + \&c.$

Second: $Z = \frac{1}{(1-z)^2}\times\{1-(1-z)\}^{-2} = \frac{1}{(1-z)^2} + \frac{2}{1-z} + \&c.;$

therefore, by this theorem $\left(\frac{1}{z^2-z}\right)^2 = \frac{1}{z^2} + \frac{1}{(1-z)^2} + \frac{2}{z} + \frac{2}{1-z}$

Example (2). Decompose $Z = \frac{1}{z^n.(z-1)^n}$ n even.

$$Z = \frac{1}{z^n}.(1-z)^{-n} = \frac{1}{z^n} + n.\frac{1}{z^{n-1}} + \frac{n(n+1)}{1\,.\,2}.\frac{1}{z^{n-2}} + \ldots\ldots$$
$$\frac{n(n+1)(n+2)\,.\,.\,.\,(2n-2)}{1\,.\,2\,.\,.\,3\,.\,.\,.\,(n-1)}\cdot\frac{1}{z} + \&c.$$

and $$Z = \frac{1}{(1-z)^n}\{1-(1-z)\}^{-n} = \frac{1}{(1-z)^n} + n\,.\,\frac{1}{(1-z)^{n-1}} +$$
$$\frac{n(n+1)}{1\,.\,2}\cdot\frac{1}{(1-z)^{n-2}} + \ldots.$$

Therefore, $$Z = \left\{\frac{1}{z^n} + \frac{1}{(1-z)^n}\right\} + n\left\{\frac{1}{z^{n-1}} + \frac{1}{(1-z)^{n-1}}\right\} +$$
$$\frac{n(n+1)}{1\,.\,2}\left\{\frac{1}{z^{n-2}} + \frac{1}{(1-z)^{n-2}}\right\} + \&c.$$

the last term being $\frac{n(n+1)\,(n+2)\ldots.2(n-1)}{1\,.\,2\,.\,3\,.\,.\,.\,.\,(n-1)}\cdot\left\{\frac{1}{z}+\frac{1}{1-z}\right\}$

Example 3. The same when n is odd. The same method is easily applied.

The decomposition contained in these two examples is of use in finding the complete solution of a remarkable differential equation, of which Laplace only used a particular solution. (*Vide* 3rd Memoir on the Inverse Method of Definite Integrals, *Camb. Phil. Trans.* Vol. VI.)

Corollary. Put $z = \frac{1}{2}$, then when n is a positive integer,

$2^{n-1} = 1 + \frac{n}{2} + \frac{n(n+1)}{2\,.\,4} + \frac{n(n+1)\,(n+2)}{2\,.\,4\,.\,6}\ldots.$ n terms. And the same series continued to infinity would be 2^n, that is the first n terms are exactly one-half the sum of the whole series. For the remaining half we have

$$2^{n-1}=\frac{n(n+1)\,(n+2)\ldots(2n-1)}{2\,.\,4\,.\,6\,.\;\;.\;\;.\;\;2n}$$
$$\left\{1+\frac{2n}{2n+2}+\frac{2n(2n+1)}{(2n+2)\,(2n+4)}+\&c.\right\}$$

Example 4. Decompose $\dfrac{1}{z^m(1-z)^n}$ and $\dfrac{z^m}{(1-z)^n}$.

Example 5. Decompose $\dfrac{1}{z(z-a)^2(z-b)^3}$.

The partial fractions which result from the decomposition of any proper rational fractions are all of the form $\dfrac{A}{(a-z)^n}$, the expansion of which, viz. $\dfrac{A}{a^n}\left\{1+n\,.\,\dfrac{z}{a}+\dfrac{n(n+1)}{1.2}\,.\,\dfrac{z^2}{a^2}\;\&c.\right\}$ is a figurate series, the coefficients of $\dfrac{z}{a}$ being figurate numbers, which are distinguished by the property that the mth figurate number of any order is the sum of m figurate numbers of the next inferior order, which property is obvious if we equate the coefficients of z^m on both sides of the identity

$$(1-z)^{-(n+1)}=(1-z)^{-1}\times(1-z)^{-n}.$$

Now, since every recurring series arranged according to the powers of z is the expansion of a proper rational fraction, it follows that every recurring series may always be decomposed into figurate series; and in the case where all the roots of the denominator are unequal, these series are geometrical.

By this decomposition also it is easy to find the general term, and the sum of x terms of any recurring series.

Let $\dfrac{Aa^n{}_1}{(a-z)^n}$ be one of the partial fractions, the coefficient of z^x in the expansion of this fraction $=A\,.\,\dfrac{n.(n+1)\,(n+2)\ldots n+x-1}{1\,.\,2\,.\,3\;\;.\;\;.\;\;.\,x}\,.\left(\dfrac{1}{a}\right)^x$

$=A\,.\,\dfrac{(x+1)\,(x+2)\ldots(x+n-1)}{1\,.\,2\,.\,3\,.\;\;.\,(n-1)}\,.\left(\dfrac{1}{a}\right)^x$, and collecting the coefficient of x from each partial fraction developed, the sum will be the coefficient of x in the recurring series.

With respect to the summation of recurring series it may be effected by observing that, since

$$\frac{a^x-z^x}{a-z}=a^{x-1}+a^{x-2}z+a^{x-3}z^2+\ldots+z^{x-1}$$

we can, by taking the derived equations relative to a determine the sums of figurate series, and therefore those of recurring series which are decomposable into them.

76. Application of Recurring Series to the Solution of Equations.

Suppose A_1, A_2, $A_3\ldots A_n$ are the constants of relation of a recurring series, of which the general term is u_x, the following equation gives the case by which each term is formed from the n preceding terms:

$$u_{x+n}=A_1u_x+A_2u_{x+1}+A_3u_{x+2}+\ldots+A_nu_{x+n-1}.$$

We know, moreover, that u_xz^x is the general term of the expansion of a proper rational fraction of the form,

$$\frac{f(z)}{1-A_1z^n-A_2z^{n-1}-A_3z^{n-2}\ldots-A_nz.}$$

Let α, β, γ, &c., be the roots of the equation,

$$y^n=A_1+A_2y+A_3y^2+\ldots+A_ny^{n-1},$$

then $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$, &c., are the roots of the above denominator equated with zero, and the factors of that denominator, beside a constant, are therefore

$$z-\frac{1}{\alpha},\ z-\frac{1}{\beta},\ z-\frac{1}{\gamma}\ \text{\&c.},$$

which we shall suppose at present to be all unequal.

The fractions may be therefore written in the form

$$\frac{f(z)}{B(1-\alpha z)(1-\beta z)(1-\gamma z)\ldots\ldots},$$

which may be decomposed into partial fractions, (as above shown) or is equal to

$$\frac{C_1}{1-\alpha z}+\frac{C_2}{1-\beta z}+\frac{C_3}{1-\gamma z}+\text{\&c.};$$

where the coefficient of z^x found by expanding each of these simple fractions is evidently $C_1\alpha^x+C_2\beta^x+C_3\gamma^x+$ &c., it follows that this formula expresses the general term of the given recurring series, or

$$u_x=C_1\alpha^x+C_2\beta^x+C_3\gamma^x+\text{\&c.};$$

therefore,

$$\frac{u_{x+1}}{u_x}.=\frac{C_1\alpha.\alpha^x+C_2\beta.\beta^x+C_3\gamma.\gamma^x+\text{\&c.}}{C_1.\alpha^x+C_2.\beta^x+C_3.\gamma^x+\text{\&c.}}$$

$$=\alpha.\frac{C_1+C_2.\left(\frac{\beta}{\alpha}\right)^{x+1}+C_3.\left(\frac{\gamma}{\alpha}\right)^{x+1}+\text{\&c.}}{C_1+C_2.\left(\frac{\beta}{\alpha}\right)^{x}+C_3.\left(\frac{\gamma}{\alpha}\right)^{x}+\text{\&c.}}$$

Suppose now that α is the greatest of the roots, then $\left(\frac{\beta}{\alpha}\right)^x$, $\left(\frac{\gamma}{\alpha}\right)^x$ diminish rapidly as x increases, and we therefore find

$$\alpha=\text{Limit of }\frac{u_{x+1}}{u_x}\quad\text{when } x=\infty.$$

To converge to the greatest root of the equation

$$y^n=A_1+A_2y+A_3y^2+\ldots\ .+A_{n-1}y^n$$

assume n arbitrary numbers for the first n terms of a recurring series, of which the constants of relation are $A_1, A_2, A_3\ldots A_n$, the quotient arising by the division of any of the *formed* terms of this series by the preceding term converges to the greatest root, the more nearly as the term is more remote from the origin of the series.

Example 1. Given $y^3 = -4 + 3y^2$. (See *Euler's Introduction.*)
Assume for the 3 arbitrary numbers 0, 1, 3, and from them construct a recurring series, of which the constants of relation are $-4, 0, 3$, the terms will be

0, 1, 3, 9, 23, 57, 135, 313, 711, 1593, 3527, &c.

We have selected this example after Lagrange, to show the slow convergence to which this method is liable in the case when the greatest is a double or triple root, that is one of two, three, &c., equal roots : in the present case the roots of the equation are 2, 2, -1, and the series converges very slowly to the double root 2.

To see the reason of this we must recur to the decomposition of fractions. When there are equal roots to the denominator thus, if $\alpha = \beta$, the theory before given shows the rational fraction which generates the series of recurring coefficients, when decomposed is of the form

$$\frac{C_1}{(1-\alpha z)^2} + \frac{C_2}{1-\alpha z} + \frac{C_3}{1-\beta z} + \frac{C_4}{1-\gamma z} + \&c.$$

and in such case we have

$$u_x = \text{coefficient of } z^x = \{C_1 \,.\, (x+1) + C_2\}\, \alpha^x + C_3 \beta^x + \&c.$$

$$\frac{u_{x+1}}{u_x} = \alpha \,.\, \frac{\{C_1(x+2) + C_2\} + C_3 \,.\left(\frac{\beta}{\alpha}\right)^{x+1} \&c.}{\{C_1(x+1) + C_2\} + C_3 \,.\left(\frac{\beta}{\alpha}\right)^{x} \&c.}$$

And as x increases only by unity each term of the series, it is clear that though $\left(\frac{\beta}{\alpha}\right)^x \left(\frac{\gamma}{\alpha}\right)^x$ rapidly diminish, yet $\frac{C_1(x+2)+C_2}{C_1(x+1)+C_2}$ does no converge to unity with sufficient rapidity for practice.

Example 2. Given $y^2 = 10 + 9\,y$.

Assume 1, 5 for the two first, and with the constants of relation 10, 9 form a recurring series as follows :

1, 5, 55, 545, 5455, 54545, 545455, 5454545,

it is clear from inspection that the quotients made by dividing each term by the preceding, viz.

$$\frac{545}{55}, \frac{5455}{545}, \frac{54545}{5455}, \frac{545455}{54545}, \frac{5454545}{545455} \&c.$$

converge rapidly to 10 being alternately less and greater, the error of $\frac{u_{x+1}}{u^x}$, or its difference from 10 being always here $= \frac{\pm 5}{u_x}$.

Example 3. Given $y^3 = 1 - 10\,y - 6\,y^2$.

The constants of relation are $1, -10, -6$, but if we make $y = \frac{1}{v}$ we have

$$v^3 = 1 + 6\,v + 10 v^2,$$

when the constants are positive.

Make 0, 1, 5, the first three terms of a recurring series, in which the constants of relation are 1, 6, 10, the subsequent terms are,

$$56, 591, 6251, 66112, 699217 \ldots.$$

Now if u_x be the xth term of this series, then $\frac{u_{x+1}}{u_x}$ converges to v, and since $v = \frac{1}{y}$, therefore $\frac{u_x}{u_{x+1}}$ converges to y, thus the approximate value of y, we have $\frac{66112}{699217} = .09455147\ldots..$ which is correct to the last figure, and closer than the approximations deduced by Lagrange (Note VI., *Traite de la Résolution des Equations Numeriques*), this example has also been treated by Newton, with his method of approximation.

Let us next consider how far this method may be affected by the existence of impossible roots; for this purpose let $\beta = a + b\sqrt{-1}$, which may be put under the form $R\{\text{Cos}\,\theta + \sqrt{-1}\,\text{Sin}\,\theta\}$, $R\,\text{Cos}\,\theta = a$, $R\,\text{Sin}\,\theta = b$; therefore $R^2 = a^2 + b^2$, then by Demoivre's theorem in trigonometry we have $\beta^n = R^n\{\text{Cos}\,n\,\theta + \sqrt{-1}\,\text{Sin}\,n\,\theta\}$; hence it is clear that α must be greater than R, or α^2 must be greater than the product of a pair of conjugate imaginary roots, that this method may be successful in converging to the greatest root.

77. I will here add a simple rule for the extraction of the square roots of integers, which is founded on the principles of recurring series, and is taken from an unpublished memoir by the Author.

Let a be the difference between the given number N, and the next perfect square which is less than it.

Let b be the difference between the perfect squares, which are immediately greater, and less than N.

Take any proper fraction $\frac{u_0}{v_0}$, and form from it a second $\frac{u_1}{v_1}$, the new numerator u_1 being found by multiplying the denominator v_0 by a, and adding the numerator, thus $u_1 = av_0 + u_0$ and the new denominator by multiplying v_0 by b, and adding u_0; similarly let $\frac{u_2}{v_2}$, $\frac{u_3}{v_3}$, &c., be successively formed, then $\frac{u_x}{v_x}$ converges to the *decimal* part of $\sqrt{N}$.

For let m be the integer part of $\sqrt{N}$, then $a = N - m^2$, $b = 2m+1$, the quantities u_x, v_x, are, by hypothesis, formed successively by the equations.

$$u_{x+1} = (N - m^2)v_x + u_x \quad \ldots \quad (1)$$

$$v_{x+1} = (2m+1)v_x + u_x \quad \ldots \quad (2)$$

From equation (2) we find $\frac{u_x}{v_x} = \frac{v_{x+1}}{v_x} - (2m+1) \quad \ldots \quad (3)$

Subtract now equation 2 from that which it becomes, when $x+1$ is put for x.

Hence $v_{x+2}-v_{x+1}=(2m+1)\ (v_{x+1}-v_x)+(u_{x+1}-u_x)$ . . . (4)

But by (1) $u_{x+1}-u_x=(N-m^2)v_x$

Substitute in (4), and arrange it according to the sub-indices of v.

Therefore $v_{x+2}-2(m+1)v_{x+1}+\{(m+1)^2-N\}v_x=0$,

or v_x is the general term of a recurring series, of which the constants of relation are $-\{(m+1)^2-N\}$ and $2(m+1)$, and since the roots of the equation,

$$y^2=-\{(m+1)^2-N\}+2(m+1).y$$

are $(m+1)+\sqrt{N}$, and $(m+1)-\sqrt{N}$, of which the former is the greater; it follows, by the preceding theory, that the limit of $\frac{v_{x+1}}{v_x}=(m+1)+\sqrt{N}$; therefore, by equation (3), the limit of $\frac{u_x}{v_x}=\sqrt{(N)}-m$; now m^2 being the greatest perfect square below N, it follows that $\sqrt{N}-m$ is the decimal part of $\sqrt{N}$, from whence the theorem is evident.

Example. To find the decimal part of $\sqrt{10}$, by a series of converging fractions.

The nearest perfect squares below and above 10, are 9 and 16; therefore, $a=1$, $b=7$; therefore assume any proper fraction, add its numerator and denominator for a new numerator, and add its numerator to 7 times its denominator for a new denominator; thus a series of converging fractions will be formed, the limit of which is the decimal part of $\sqrt{10}$.

Let $\frac{1}{6}$ be the proper fraction assumed, then the converging fractions arising are $\frac{7}{43}$, $\frac{25}{154}$, $\frac{179}{1103}$, $\frac{1282}{7900}$ . . . . . if the last fraction be converted to a decimal, we find for $\sqrt{10}$ the quantity 3.162278 . . . . which is correct, except the last figure, which should be 7.

When the method of approximation to the greatest root of a given equation, by means of recurring series is used, if the equation be transformed, so that its two greatest roots may have contrary signs, the converging fractions will then be alternately greater and less than the true value; consequently if two consecutive converging fractions be reduced to decimals, the cyphers which are common to them, must certainly belong to the true root; this advantage is not possessed by many other known methods of approximation, and it can be easily found as follows:

Let u_x be the general term of the recurring series, and α, β, γ, &c., the roots of the given equation, of which the two numerically greatest are α, β, which we suppose to have contrary signs, then

$$u_x=C_1\alpha^x+C_2\beta^x+C_3\gamma^x+\text{ &c.}$$

$$\frac{u_{x+1}}{u_x}=\alpha\cdot\frac{1+c_1.\left(\frac{\beta}{\alpha}\right)^{x+1}+c_2\left(\frac{\gamma}{\alpha}\right)^{x+1}+\text{ &c.}}{1+c_1.\left(\frac{\beta}{\alpha}\right)^{x}+c_2.\left(\frac{\gamma}{\alpha}\right)^{x}+\text{ &c.}}$$

where c_1, c_2, &c., are put for $\frac{C_2}{C_1}$, $\frac{C_3}{C_1}$, &c.

$$\text{Hence } \frac{u_{x+1}}{u_x} - \alpha = \frac{c_1 \frac{(\beta-\alpha)}{\alpha}\left(\frac{\beta}{\alpha}\right)^x + c_2 \cdot \frac{\gamma-\alpha}{\alpha} \cdot \left(\frac{\gamma}{\alpha}\right)^x + \ \&c.}{1 + c_1\left(\frac{\beta}{\alpha}\right)^x + c_2 \cdot \left(\frac{\gamma}{\alpha}\right)^x + \ \&c.}$$

Now when x is great, since $\beta > \gamma$ $\left(\frac{\beta}{\alpha}\right)^x$ is very great compared with $\left(\frac{\gamma}{\alpha}\right)^x$ &c., but is itself a very small fraction, because $\alpha > \beta$; hence the difference between the converging fraction and the true root is $c_1 \cdot \frac{\beta-\alpha}{\alpha} \cdot \left(\frac{\beta}{\alpha}\right)^x$ very nearly, and consequently it is alternately positive and negative, $\frac{\beta}{\alpha}$ being necessarily negative; therefore $\frac{u_{x+1}}{u_x}$ is alternately greater and less than α.

78. The sums or differences of the corresponding terms of two recurring series, one of which has n constants of relation, and the other n', will itself be a recurring series, having $n+n'$ constants of relation.

For let u_x, v_x be the general terms of the two recurring series respectively, then if α_1, $\alpha_2 \ldots \alpha_n$ β_1, $\beta_2 \ldots \beta_n$ be respectively the roots of the denominators of their generating fractions, when equated with zero, we have

$$u_x = C_1\alpha_1^x + C_2\alpha_2^x + C_3\alpha_3^x + \ \&c.$$
$$v_x = B_1\beta_1^x + B_2\beta_2^x + B_3\beta_3^x + \ \&c.$$

where B_1, B_2, &c. C_1, C_2, &c., are constants,

therefore $u_x \pm v_x = C_1\alpha_1^x + C_2\alpha_2^x + \ldots + C_n\alpha_n^x \pm B_1\beta_1^x \pm B_2\beta_2^x \pm \ldots B_n\beta_n^x$, which is obviously the general term of another recurring series, the denominator of the generating fraction of which is the product of the two denominators, and contains $n+n'$ constants. The same would obviously be true relative to *figurate* recurring series, if any of the denominators contained equal roots.

The products of the corresponding terms of two recurring series, one of which has n constants, and the other n' constants of relation, different from the former, is itself a recurring series, having nn' constants of relation.

Again, if u_x be the general term of a recurring series of n constants, $u_x\ u_{x+1}$ is the general term of another recurring series, having $\frac{n(n+1)}{2}$ constants of relation; these properties are easily proved in the same manner as the first.

79. In the *Expose Synoptique*, which precedes the posthumous treatise of Fourier, *Analyse des Equations*, there are some remarks on recurring series, which contain original theorems relative to the discovery of the different roots of equations by this method. No demonstration of these theorems has yet been published within the knowledge

of the author of this treatise; we give here the substance of these remarks, merely adapting the notation to our own.

Suppose that we have formed the primitive recurring series, which is directly derived from the coefficients of the proposed equation, or the constants of relation, and from first terms which are arbitrary. Let u_x represent the xth term of this series, and let α', α'', α''', &c., be the roots of the proposed, written in the order of their magnitude (abstraction being made of the sign), if there are imaginary roots, their magnitude is estimated by the square root of the product of a conjugate pair, as we have already seen.

If the root α', which occupies the first place, is real, we can approximate to it indefinitely by dividing each term of the recurring series by the preceding; this has been also proved, but it only gives one root.

To determine the following roots, take four consecutive terms, as u_1, u_2, u_3, u_4, form then the product $u_1.u_4$, of the extreme terms, and subtract from it the product $u_2.u_3$, of the two mean terms; write the remainder, $u_1 u_4 - u_2 u_3$, below the first series, and perform in the same way this operation for the four consecutive terms u_2, u_3, u_4, u_5, and then for the next four, u_3, u_4, u_5, u_6, and so on.

We shall have thus a second series, v_0, v_1, v_2, v_3, &c., derived from the first, of which we may express thus the general term, $v_x = u_x\, u_{x+3} - u_{x+1}\, u_{x+2}$.

This second series is recurring, and the limit of the series of quotients $\frac{v_2}{v_1}$, $\frac{v_3}{v_2} \cdots \frac{v_{x+1}}{v_x}$, is the sum $\alpha' + \alpha''$ of the two first roots of the proposed, and as the first α' is known by a preceding operation, the value of the second root α'' will thus become known also.

If instead of choosing four consecutive terms of the first series, we only take three consecutive terms u_1, u_2, u_3, and if from the product $u_1.u_3$, of the extreme terms, the square of the mean term u_2 be subtracted, these remainders will generate another series, of which the general term is $u_x\, u_{x+2} - u_{x+1}^2$.

This series is also recurring, and the quotients arising from the division of each term, by that immediately preceding, converge towards $\alpha'.\alpha''$, that is the product of the two first roots.

And in like manner rules may be given for finding the sum, the sum of the products two and two, and the absolute product of the first three roots, and so on. From what has been said we can form,

First, A recurring series, from which the approximate values of the root α' are known.

Secondly, A series of quotients which give the value of the product $\alpha'\,\alpha''$.

Thirdly, A third series, which gives the value of the product $\alpha'\,\alpha''\,\alpha'''$ of the three first roots, and so on.

If the first root is imaginary, that is to say, if the product of two conjugate imaginary roots exceeds the square of each real root, the first series will give no result, the series of continued quotients will be divergent and vague, as remarked by Euler [and the reason of this is easily perceived from the form of α'^x, viz., $\Pi^x(\text{Cos}\, x\,\phi + \sqrt{-1}\,\text{Sin}\, x\,\phi)$ the trigonometrical functions creating a species of periodicity]. But in the same case the second series of quotients is convergent, and the

limit of these continued quotients is the real product $\alpha' \alpha''$ of the conjugate imaginary roots.

If the third root is real, the third series of quotients is convergent. The contrary happens when the third root is imaginary, but then the fourth series of quotients corresponding to $\alpha' \alpha'' \alpha''' \alpha^{\text{v}}$ is necessarily convergent; thus two consecutive series may both give convergent, but cannot *both* give divergent results.

From these theorems it follows that to know in all cases the roots of a proposed equation, it suffices to form series relative to the successive products and to the successive sums of the roots. Thus we shall find the approximate values of all the real roots, and for each imaginary root, the real part and the coefficient of $\sqrt{-1}$; this is the most extended use which has been made of the method of recurring series for the solution of equations.

Fourier concludes with observing, that this method in practice cannot be deemed sufficiently expeditious; the examples given by Euler are ingeniously chosen, but that mode of approximation requires in general too much calculation; we only consider this question in a theoretical point of view. The properties announced, he adds, are *incomparably* more general than those known to the first inventors of those series, and the authors who have since treated on them, stating that his object was to complete one of the principal elements of Algebraical Analysis.

Though M. Fourier has opened here new views with respect to the application of recurring series to numerical equations, yet, by some singular oversight, the preceding theorems, with the exception of those referring to the continued products of the roots, are undoubtedly incorrect; the transformations above indicated give recurring series, and if $v_x = C\alpha^x + C'\beta^x + C''\gamma^x +$ &c., then $\frac{v_{x+1}}{v_x}$, will, as we have seen, converge to a, the greatest of the quantities, α, β, γ, &c., if it converge at all, now none of the preceding transformations will introduce a sum of two roots in the general expression for the transformed recurring series, for a term such as $(\alpha+\beta)^x = \alpha^x + x\alpha^{x-1}\beta +$, &c., could not be deduced from the primitive recurring series, without in some way introducing x in the transformation, which is not done in Fourier's rules, but a combination of two recurring series, in the way of a quotient, *which does not generate a recurring series*, is able to give the simple sums of roots, or the sums of symmetrical functions of a certain number of roots. It will be here the most satisfactory course to investigate the analytical expressions on which those rules are founded, and then give a correct rule for finding the sum of two roots by means of recurring series.

Let α, β, γ, &c., be the roots of an algebraical equation $\phi(x)=0$, and let u_1, u_2, u_3, $u_4 \ldots u_x$, &c., be the terms of a recurring series, formed by means of its coefficients, and certain arbitrary quantities, as before shown, so that

$$u_x = C\alpha^x + C'\beta^x + C''\gamma^x + \text{\&c.}$$

and consider first the value of the product $u_{x+n}\, u_{x+n'}$, in which we suppose n' not less than n.

This product will consist of two parts, in one of which the simple roots α, β, γ, &c., will be raised to the power $2x$, with certain coefficients, and the other of the rectangles of the roots $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$, &c., raised to the power x, with different coefficients.

Thus $u_{x+n} = C\alpha^n . \alpha^x + C'\beta^n . \beta^x + C''\gamma^n . \gamma^x +$ &c.

$u_{x+n'} = C\alpha^{n'} . \alpha^x + C'\beta^{n'} . \beta^x + C''\gamma^{n'} . \gamma^x +$ &c.

Therefore $u_{x+n}u_{x+n'} = C^2 . \alpha^{n+n'} . \alpha^{2x} + C'^2\beta^{n+n'} . \beta^{2x} + C''^2\gamma^{n+n'}\gamma^{2x} +$ &c.

$+ CC'(\alpha^n\beta^{n'} + \beta^n\alpha^{n'})(\alpha\beta)^x + CC''(\alpha^n\gamma^{n'} + \gamma^n\alpha^{n'})(\alpha\gamma)^x +$ &c.

The terms which involve the simple powers of the roots are therefore invariable, if $n+n'$ be constant, and they may be made to disappear by subtraction, thus if $m+m'=n+n'$, then

$$\begin{aligned} u_{x+n}u_{x+n'} - u_{x+m}u_{x+m'} = {} & CC'(\alpha^n\beta^{n'} + \alpha^{n'}\beta^n - \alpha^m\beta^{m'} - \alpha^{m'}\beta^m)\ (\alpha\beta)^x \\ & + CC''(\alpha^n\gamma^{n'} + \gamma^n\alpha^{n'} - \alpha^m\gamma^{m'} - \alpha^{m'}\gamma^m)\ (\alpha\gamma)^x \\ & + \text{\&c.} \end{aligned}$$

which, for abridgment, may be written $c(\alpha\beta)^x + c'(\alpha\gamma)^x + c''(\beta\gamma)^x +$ &c.

Now if we represent this remainder by v_x we find

$$\frac{v_{x+1}}{v_x} = \alpha\beta . \frac{1 + \frac{c'}{c}\left(\frac{\gamma}{\beta}\right)^{x+1} + \left(\frac{c''}{c}\right)\left(\frac{\gamma}{\alpha}\right)^{x+1} + \left(\frac{c'''}{c}\right)\left(\frac{\gamma}{\alpha} . \frac{\delta}{\beta}\right)^{x+1} + \text{\&c.}}{1 + \frac{c'}{c}\left(\frac{\gamma}{\beta}\right)^x + \frac{c''}{c} . \left(\frac{\gamma}{\alpha}\right)^x + \text{\&c.}}$$

which therefore converges to $\alpha\beta$ as x increases, α, β, being the two greatest roots.

In the case of equal roots, the convergence is more slow, as c', c'', &c., are then algebraical functions of x.

Thus let $n=0$, $n'=2$, $m=m'=1$, then the condition $m+m'=n+n'$, is satisfied, and we have $v_x = u_x u_{x+2} - u_{x+1}^2$, and therefore Fourier's rule for the *product* of the two first roots is correct, and hereby demonstrated.

Again, if $n=0$, $n'=3$, $m=1$, $m'=2$, and therefore $n+n'=m+m'$, we have $v_x = u_x u_{x+3} - u_{x+1}u_{x+2}$, and as we have found that $\frac{v_{x+1}}{v_x}$ changes to $\alpha\beta$, Fourier's rule, which says that it converges towards $\alpha+\beta$, is evidently incorrect, and it is obvious that only products, and not sums, can be obtained by means of a single recurring series, of which the general term v_x is obtained by the means above used.

But the quotient of the general terms of two recurring series, which does not itself exist as the general term of any recurring series, is susceptible of giving *sums* of the roots, or of the symmetrical functions of a certain number of roots, as its limit, when x is infinite.

For with the same notation let $\nu+\nu'=\mu+\mu'$, and let

$$V_x = u_{x+\nu}u_{x+\nu'} - u_{x+\mu}u_{x+\mu'}$$

then

$v_x = CC'(\alpha^n\beta^{n'} + \alpha^{n'}\beta^n - \alpha^m\beta^{m'} - \alpha^{m'}\beta^m)\ (\alpha\beta)^x + c'(\alpha\gamma)^x + c''(\beta\gamma)^x +$ &c.

$V_x = CC'(\alpha^\nu\beta^{\nu'} + \alpha^{\nu'}\beta^\nu - \alpha^\mu\beta^{\mu'} - \alpha^{\mu'}\beta^\mu)\ (\alpha\beta)^x + e'(\alpha\gamma)^x + e''(\beta\gamma)^x +$ &c.

where e', e'', &c., represent constants, formed similarly with c', c'', &c.

Hence, $\alpha\,\beta$ being always the two first (when real, greatest) roots, we have

$$\text{Limit of } \frac{v_x}{V_x}=\frac{\alpha^n\beta^{n'}+\alpha^{n'}\beta^n-\alpha^{m'}\beta^m-\alpha^m\beta^{m'}}{\alpha^\nu\beta^{\nu'}+\alpha^{\nu'}\beta^\nu-\alpha^{\mu'}\beta^\mu-\alpha^\mu\beta^{\mu'}}$$

Let $n=0$, $n'=3$, $\nu=0$, $\nu'=2$, $m=1$, $m'=2$, $\mu=1$, $\mu'=1$; then $v_x=u_x u_{x+3}-u_{x+1}u_{x+2}$, $V_x=u_x u_{x+2}-u_{x+1}{}^2$, and the value of this general expression is then $\frac{\alpha^3+\beta^3-\alpha^2\beta-\alpha\beta^2}{\alpha^2+\beta^2-2\alpha\beta}=\alpha+\beta$, from whence this rule will follow.

In the recurring series, formed with the scale of constants, given by the coefficients of a proposed equation, take four consecutive terms, and from the product of the extremes subtract the product of the mean terms.

Omitting the last term of the four, from the product of the extremes of the other three, subtract the square of the mean.

Divide the former remainder by the latter, the quotient will converge to the sum of the two greatest roots when real, and when the two first roots are imaginary, to double the real part of either imaginary.

This rule enabling us to find $\alpha+\beta$ approximatively, and either $\frac{v_{x+1}}{v_x}$ or $\frac{V_{x+1}}{V_x}$, converging to $\alpha\beta$, as has been proved, we can thus find α and β by the formula $\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\}=\frac{\alpha+\beta}{2}\pm\sqrt{\left\{\left(\frac{\alpha+\beta}{2}\right)^2-\alpha\beta\right\}}$

No other easily practicable method is yet known for converging to the real and imaginary parts of impossible roots; but, having thus obtained their first approximate values, we can find them as exactly as we please by T. Simpson's extension of Newton's method of approximation.

EXAMPLES.

(1.) Given $x^2=6x-10$.

Assume 1, 2, for the first two terms of a recurring series, of which the constants of relation are -10 and 6;

$$1,\ 2,\ 2,\ -8,\ -68,\ -328,\ \&c.$$

Take the four last terms, and from the product of the extremes -656 subtract the product of the means 544; the remainder is -1200.

Again (omitting the last term), from the product of the extremes -136, take the square of the mean 64, remainder $=-200$. Divide the former remainder by this, the quotient is 6, which is the sum of the roots exactly, and its one-half 3 is the real part of the imaginary roots.

From the product 2624 of the last term and last but two, subtract 4624, the square of the last but one, and divide the remainder -2000 by the former corresponding remainder -200; the quotient $+10$ is the product of the roots, or the sum of the squares of the real part, and of the coefficient of the imaginary part; this coefficient is therefore only unity, and consequently the roots are $3\pm\sqrt{-1}$; this example tests the accuracy of our rule, and of Fourier's correct rule for the product of the roots.

Example (2.) $x^3=7x-6$

Assume for first terms 0, 1, 2, scale of constants -6, 7, 0

$$7,\ 8,\ 37,\ 14,\ 211,\ \&c.$$

Now all the roots here are real, yet, if we divide each term by the preceding, the quotient, which should converge to the greatest root -3, does not until the series is continued so far that the terms shall be alternately positive and negative, but if we apply the rules for the sum and product of the two greatest, we find respectively $-.9$ and -6.05, the correct values being -1 and -6, thus the want of convergence, arising from badly assuming the first terms, may be greatly corrected by the application of these rules.

(80.) When the ratio of the first root, or first product of roots, to any other, is such that the sum of the squares of its real part and of the coefficient of its imaginary part is unity, this method will fail: thus, putting $u_x = C\alpha^x + C'\beta^x + \&c. = \alpha^x\left\{C + C'\left(\frac{\beta}{\alpha}\right)^x + \&c.\right\}$ then if $\frac{\beta}{\alpha}$ be of the form above mentioned, it may be written $\text{Cos}\,\theta + \sqrt{-1}.\ \text{Sin}\,\theta$; and therefore $\left(\frac{\beta}{\alpha}\right)^x$ will be periodical instead of converging to zero, this inconvenience may be remedied by a transformation of the equation proposed, which shall increase all its roots by a constant.

Other inconveniences, as Lagrange remarked, may be avoided, by forming the first terms according to the law for the sums of the powers of the roots; the subsequent terms will then be also similar sums of the powers expressed by the number or place of such term in the recurring series, and the nature of the derived recurring series will be then easily and accurately known.

Thus $u_x = \alpha^x + \beta^x + \gamma^x + \&c.$

$$u_x u_{x+3} - u_{x+1}u_{x+2} = (\alpha-\beta)^2(\alpha+\beta)(\alpha\beta)^x + (\alpha-\gamma)^2(\alpha+\gamma)(\alpha\gamma)^x + (\beta-\gamma)^2(\beta+\gamma)(\beta\gamma)^x, \&c.$$

$$u_x u_{x+2} - u_{x+1}^2 = (\alpha-\beta)^2\,(\alpha\beta)^x + (\alpha-\gamma)^2\,(\alpha\,\gamma)^x + (\beta-\gamma)^2\,(\beta\,\gamma)^x + \&c.$$

and it is, by inspection of these values, easily seen, that the approximation will be most favourably conducted, when γ, δ, &c., being small compared with α, β, $\alpha-\gamma$, $\beta-\gamma$, &c., are not great compared with $\alpha-\beta$; when α, β, are real, it is an advantage if their signs be contrary.

It would not be difficult to extend these rules to comprise the combinations of three or more roots, but the numerical calculations greatly multiply for the formation of the proper series, at the same time the convergence becomes much more slow, because the number of separate parts which compose the general term of the derived recurring series, from n, which it was in the primitive, increases to $\frac{n(n-1)}{2}$, $\frac{n(n-1)(n-2)}{2\,.\,3}$, &c., all of which parts, with the exception of *one* part, being rejected as small compared with it, the error arising evidently increases with the number of rejected terms. Besides the methods here explained are sufficient to find successively all the real and imaginary roots, by separating those which are found.

(81.) The early analysts bestowed much labour in the invention of methods for finding the simple and quadratic divisors of equations when rational; this branch of research is nearly useless, for in practice they seldom are rational, and, when they are so, the easy methods now existing for determining the limits of the roots of equations reduce the discovery of such divisors to a few trials.

(82.) Newton's Method of Approximation.

Let a be an approximate value to a root of the equation $\phi(x)=0$, and $a+h$ the correct value; then

$$\phi(a+h)=\phi(a)+\phi'(a)\,.\,h+\phi''(a)\,.\,\frac{h^2}{1.2}+\ \&c.=0$$

Now h being, by supposition, very small, the terms of this series which contain higher powers of h than the first are also very small compared with it; if therefore $\phi''(a)$, $\phi'''(a)$, &c., are not very great compared with $\phi'(a)$, we have very nearly $\phi(a)+h\phi'(a)=0$, or $h=-\dfrac{\phi(a)}{\phi'(a)}$ nearly. As we have not obtained the accurate value of h, denote this result by h_1, then $h_1=h$ very nearly, and $a+h_1$ is therefore very near the sought root; let $a+h_1=a_1$, and the same reasoning shows that a_1+h_2 or a_2 is a closer approximation to this root, when $h_2=-\dfrac{\phi(a_1)}{\phi'(a_1)}$, and by repeating this process we may converge in most cases very rapidly to the root required; $a_1=a-\dfrac{\phi(a)}{\phi'(a)}$, $a_2=a_1-\dfrac{\phi(a_1)}{\phi'(a_1)}$, $a_3=a_2-\dfrac{\phi(a_2)}{\phi'(a_2)}$; if therefore we put $F(a)=a-\dfrac{\phi(a)}{\phi'(a)}$, we have

$$a_1=F(a)\quad a_2=F(a_1)=F.F(a)=F^2(a),\ a_3=F(a_2)=F.F.F(a)=F^3(a)\ \&c.$$

we thus generate a *continued* function, converging to the root of the proposed equation $\phi(x)=0$.

Conversely, suppose α to be the ultimate value, to which converge a, $F(a)$, $F^2(a)$, $F^3(a)$, &c., then when n is infinite we have

$$F^n(a)=\alpha \qquad F^{n+1}(a)=\alpha.$$

therefore $\quad F(\alpha)=\alpha$

But $\quad F(\alpha)=\alpha-\dfrac{\phi(\alpha)}{\phi'(\alpha)}$

therefore $\dfrac{\phi(\alpha)}{\phi'(\alpha)}=0$; now if x_1, x_2, x_3, &c., be the roots of the proposed equation, we have

$$\frac{\phi'(\alpha)}{\phi(\alpha)}=\frac{1}{\alpha-x_1}+\frac{1}{\alpha-x_2}+\frac{1}{\alpha\ \ x_3}\ \&c.=\text{infinity}.$$

Therefore a must be equal to one of the roots x_1, x_2, x_3, &c., consequently this series of repeated functions, when convergent towards a limit, determines necessarily one root of the equation. Such are the principles on which this easy method of approximation is founded; it is obvious, however, from the nature of both the direct and converse processes above pursued, that certain circumstances are necessary to ensure its success.

(83.) By the application of Sturm's theorem (Art. 21) we can find two limits, a, b, of a single root of the equation $\phi(x)=0$, and we may suppose them, by subdividing their interval, to be taken sufficiently near each other, that no root of the equations $\phi'(x)=0$, $\phi''(x)=0$, may be included between them; this may always be effected, unless either of the latter equations have one or more roots in common with the proposed; and if there is reason to suspect that such a relation exists, it is only necessary to seek the greatest common divisor of the two functions, and proceed then according to the general rule to find the root corresponding to this common factor; these particular cases may therefore be excluded in the following considerations:

$$\text{Since } \phi(x+h) = \phi(x)+h\phi'(x)+\frac{h^2}{1.2}\cdot\phi''(x)+ \&\text{c.}$$

and by taking h sufficiently small, the terms after the second term will not influence the sign of the quantity added to $\phi(x)$, and $\phi'(x)$ never changes its sign from $x=a$ to $x=b$, therefore $\phi(x)$ is always increasing or always diminishing between those limits, according as that sign is positive or negative, and the same remark must hold for $\phi'(x)$, since $\phi''(x)$ does not vanish in the interval from $x=a$ to $x=b$.

Let this interval, $a-b$, be divided into a very great number n of parts, that is, let $h=\dfrac{a-b}{n}$.

$$\text{then } \phi(b+h) = \phi(b)+h\phi'(b)+\frac{h^2}{2}\cdot\phi''(b)+ \&\text{c.}$$

$$\text{and} \qquad \phi(b) = \phi(b+h)-h\phi'(b+h)+\frac{h^2}{2}\cdot\phi''(b+h)- \&\text{c.}$$

therefore

$$\frac{\phi(b+h)-\phi(b)}{h}=\phi'(b)+\frac{h}{2}\cdot\phi''(b)+ \&\text{c.}=\phi'(b+h)-\frac{h}{2}\phi''(b+h)+ \&$$

Now since $\phi''(x)$ does not change sign from $x=b$ to $x=a$, therefore $\phi''(b)$, $\phi''(b+h)$ have the same signs, and since h is very small, $\dfrac{\phi(b+h)-\phi(b)}{h}$ is necessarily between $\phi'(b)$ and $\phi'(b+h)$, but $\phi'(x)$ has a continuous increase or decrease from b to $b+h$, therefore $\dfrac{\phi(b+h)-\phi(b)}{h}$, must be exactly equal to $\phi'(b_1)$, where b_1 is some quantity between b and $b+h$.

Again, b_2 being between $b+h$ and $b+2h$, b_3 between $b+2h$, and $b+3h$, we must have

$$\frac{\phi(b+2h)-\phi(b+h)}{h} = \phi'(b_2)$$

$$\frac{\phi(b+3h)-\phi(b+2h)}{h} = \phi'(b_3)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\phi(a)-\phi(b+\overline{n-1}\,.\,h)}{h} = \phi'(b_n)$$

and taking the sum of all these differences, we find

$$\phi(a)-\phi(b) = h\{\phi'(b_1)+\phi'(b_2)+\phi'(b_3)+\ldots\ldots+\phi'(b_n)\}$$

But $\phi'(x)$ continually increases, or continually diminishes from $x=b_1$ to b_n, which are between a and b; therefore $\phi(a)-\phi(b)$ is necessarily between $nh\,\phi'(b_1)$ and $nh\,.\,\phi'(b_n)$ or $\dfrac{\phi(a)-\phi(b)}{a-b}$ is between $\phi'(b_1)$ and $\phi'(b_n)$, and is therefore exactly equal to $\phi'(\alpha)$, α being some quantity between b_1 and b_n, or, taking more extended limits, between a and b.

(84.) To apply this, suppose a greater and b less than a sought root of the equation $\phi(x)=0$, and such as not to include any root of the equations $\phi'(x)=0$ $\phi''(x)=0$ between them, let e, e_1, be the errors, that is, let $a-e=b+e_1$ be the true root, then

$$\phi(a-e) = \phi(a)-e\,\phi'(\alpha) = 0$$
$$\phi(b+e_1) = \phi(b)+e_1\phi'(\beta) = 0$$

Where α is between a and $a-e$, or x, and β between b and $b+e_1$, or x, and therefore both α and β are certain quantities between a and b,

$$\text{therefore } x = a-e = a-\frac{\phi(a)}{\phi'(\alpha)}$$

$$= b+e_1 = b-\frac{\phi(b)}{\phi'(\beta)}$$

Let $\phi'(a)$ be the greatest of the two $\phi'(a)$, $\phi'(b)$; then it will be greater than $\phi'(\alpha)$, $\phi'(\beta)$, though with the same sign, and $\phi(a)$, $\phi(b)$, have obviously contrary signs; therefore, if $\phi'(a)$ be written in the above expressions for $\phi'(\alpha)$, $\phi'(\beta)$, the correcting fractions for e, e_1, will both be diminished, and therefore $a-\dfrac{\phi(a)}{\phi'(a)}$, $b-\dfrac{\phi(b)}{\phi'(a)}$, are nearer limits to the sought root than a and b, and are, one greater, the other less, than it.

Calling these new limits a', b', we find, in the same way, closer limits, $a''=a'-\dfrac{\phi(a')}{\phi'(a')}$, $b''=b'-\dfrac{\phi(b')}{\phi'(a')}$, and so on; and if we converge thus to the true root from quantities above and below it, it is obvious that the digits common to the two approximations belong strictly to

the true root; thus this method of approximation will proceed with accuracy and rapidity.

Example. $x^3-2x-5=0$.

$$\phi(x)=x^3-2x-5\,;\ \phi'(x)=3x^2-2\,;\ \phi''(x)=6x.$$

Divide now $\phi(x)$ by $\phi'(x)$, and continue the division in the manner of finding the greatest common measure of these quantities, observing to change the sign of the remainders before they are used as new divisors, thus:—

$$
\begin{array}{lrl}
 & 3x^2-2)\ 3(x^3-2x-5)\ (x & \\
 & 3x^3-2x & \\
 & \overline{-4x-15} & \\
\text{Second divisor} \ldots\ldots & 4x+15)\ 4(3x^2-2)\ (3x & \\
 & 12x^2+45x & \\
 & \overline{-45x-8} & \\
\text{Multiply by } 4 \ldots\ldots\ldots & -180x-32\ (-45 & \\
 & -180x-675 & \\
 & \overline{+643} & \\
\text{Third divisor} \ldots\ldots\ldots\ldots & -643 &
\end{array}
$$

The series formed by the given function and the successive divisors will in this instance be x^3-2x-5; $3x^2-2$; $4x+5$; -43. The signs of the first terms in these expressions are $+$, $+$, $+$, $-$, which have but one alternation of sign; therefore the equation has a pair of impossible roots. Put successively 2, 3, for x in the same functions, they become in the first case -1, $+10$, $+13$, -43, and in the second $+16$, $+25$, $+17$, -43; the former substitution giving one alternation of sign more than the latter, it follows that the equation $x^3-2x-5=0$ has one, and only one, real root between 2 and 3.

Also, it is easily seen that neither of the equations $3x^2-2=0$, $6x=0$, have any root between the same limits; these numbers will therefore serve to commence the approximation.

Put therefore $x=3-e=2+e_1$;

then $$\phi(3-e)=\phi(3)-e\phi'(\alpha)=0,$$

where α is between x and 3, and $\phi'(x)$ increases between these limits; therefore $\phi'(\alpha)<\phi'(3)$, or 25; and since $\phi(3)=16$, therefore $\frac{16}{25}<e$; therefore $x<3-\frac{16}{25}$, that is, 2.36 is a superior limit nearer than 3 to the true root.

Similarly, $\phi(2+e_1)=\phi(2)+e_1\phi'(\beta)$. β lying between 2 and x, we have $\phi'(\beta)<\phi'(3)$, or 25; therefore e_1 is greater than $\frac{1}{25}$, or $x>2.04$, a nearer inferior limit than 2.

Though the limits 2.04, 2.36, are nearer than those first assumed, yet, as the digits in the first place of decimals differ by more than a decimal unit, we shall readily obtain a nearer superior limit by writing 2.1, 2.2, 2.3, for x in $\phi(x)$. The first of them which renders $\phi(x)$ positive must be a superior limit, since there is but one real root be-

tween 2 and 3, and $\phi(x)$ can only change its sign by passing through zero. Now, when 2.1 is put for x, we find $\phi(x)=.061$, which is positive.

We shall therefore recommence the operation with the limits 2 and 2.1.

$$\phi(2.1)=9,261-4.2-5=.061$$
$$\phi(2)=8-4-5=-1$$
$$\phi'(2.1)=13.23-2=11.23;$$

therefore $$x<2.1-\frac{61}{11230}, \text{ or } 2+\frac{1062}{11230},$$

and $$x>2+\frac{1000}{11230}.$$

These limits of x differ only by $\frac{62}{11230}$; but the superior limit has a much nearer approximation to the root than the lower, the divisor of the correction in the latter being foreign to it, and borrowed from the superior limit; from this, converted into decimals, we have $x<2.09456$, which is remarkably close as a second approximation, the true value, given by repeated processes in Fourier's 'Analyse,' being

$$2.09455.14815.42326.59148.23865.40579.30\ldots\ldots$$

This example was first given as an instance of his method by Newton, was afterwards solved differently by Lagrange, 'Equations Numériques,' and the cause of the rapid convergence by Newton's method pointed out 'Note V.,' and finally was selected by Fourier to exemplify his improvements in the theory of this approximation. He has, however, indulged in numerical calculations which were unnecessary; for the sign of $\phi(x)$, when any approximation was put for x, would show whether that value was above or below the true one, because only one real root lies between 2 and 3; and since 2 makes $\phi(x)$ negative, and 3 makes it positive, the sign alone for intermediate values would suffice to determine whether they were superior or inferior limits. We may observe that the logarithmic method, which gives the same result as Lagrange's series, would in this example give

$$x=\frac{5}{2}-\frac{5^3}{2^4}+\frac{6}{1.2}\cdot\frac{5^5}{2^7}-\frac{8.9}{2.3}\cdot\frac{5^7}{2^{10}}+\&c.,$$

a divergent series corresponding to the imaginary roots.

It is important to consider the degree of accuracy attained by this method of approximation.

Suppose, as before,* that a, b are the superior and inferior limits of the root to which we wish to approximate, containing between them no other root of the equation $\phi(x)=0$, nor any root of the equations $\phi'(x)=0$, $\phi''(x)=0$; and let $a-b=h$, $a'=a-\frac{\phi(a)}{\phi'(a)}$, $b'=b-\frac{\phi(b)}{\phi'(a)}$, $h'=a'-b'$; then a', b' are the limits obtained by the first approximation, the accuracy of which may be judged by the relation subsisting between h' and h.

Substitute for b its value $a-h$; then $\phi(b)=\phi(a)-\phi'(a).h+\phi''(\gamma).\frac{h^2}{1.2}$, where it may be seen, by similar reasoning to that we have already employed, that γ is some quantity between a and b; hence we obtain

$$b'=a-h-\frac{\phi(a)}{\phi'(a)}+h-\frac{\phi''(\gamma)}{2\phi'(a)}.h^2;$$

therefore $$a'-b'=h'=\frac{\phi''(\gamma)}{2\phi'(a)}.h^2.$$

Now, though the quantity γ and therefore $\phi''(\gamma)$ is not accurately known, yet we can easily determine a known quantity greater than the latter—for instance, $\phi''(a)$—if $\phi'''(a)$ is positive, or $\phi''(b)$ in the contrary case, on the allowable supposition that $\phi'''(x)$ has no root in the interval from b to a; and dividing this quantity greater than $\phi''(\gamma)$ by $2\phi'(a)$, if C be the quotient, we necessarily have $h'<Ch^2$; and, as $\frac{\phi''(\gamma)}{\phi'(a)}$ would diminish for nearer approximations (since $\phi''(\gamma)=\phi''(a)-\phi'''(\delta).k$, where δ is between a and b, or to $\phi''(b)+\phi'''(\varepsilon).l$ and $\phi'''(\delta)$, $\phi'''(\varepsilon)$, must have, one the same sign as $\phi'(a)$, the other the contrary sign; $k=a-\gamma$, $l=\gamma-b$); therefore, when C is properly determined, if h', h'', h''', &c., be the differences of the new limits successively obtained, we have $h'<Ch^2$, $h''<Ch'^2$, $h'''<Ch''^2$, &c.; wherefore the rapidity of the approximation may be compared to that of a descending hypergeometrical progression; for if $h'=\frac{A}{10^n}$, then $h''<\frac{CA^2}{10^{2n}}$, the number of places of correct decimals, excepting the influence of C, which is always the same, being thus doubled at each operation. Such great convergence makes this mode of approximation as valuable as it is sure.

When $a-e$ is put for x, if, instead of determining e approximately by the equation $\phi(a)-e\phi'(a)=0$, we take the more accurate equation $\phi(a)-e\phi'(a)+e^2\frac{\phi''(a)}{1.2}$, then the verification will subsist in the equation $\phi(a-e)=0$ to the second power of e inclusive; and if h be the difference of new limits obtained in this manner, the same considerations show that the diminution of the successive errors is still more rapid, or may be expressed by $h'=Ch^3$, $h''=Ch'^3$, &c.; but this extension of the Newtonian approximation, requiring at each operation the extraction of a root, is less simple, and therefore less fit for use, than that which we have explained.

For the great improvement of Newton's approximation by means of two limits, which include in their interval all the roots of the first and second derived functions, we are indebted to Fourier; but his method would have been incomplete without the assistance of Sturm's theorems, which show definitively the existence or non-existence of roots in given intervals. For the employment of parabolic curves, however it may tend to illustrate the subject as an application, must be regarded as foreign to it when used as a mode of investigation; and in reality it furnishes only a vicious circle, since nothing can be deduced from a curve which is not implied in the equation from which it is traced.

(85.) Without entering on the general subject of elimination, we

shall notice Simpson's extension of the Newtonian method of approximation.

Let $\phi(x, y)=0$, $F(x, y)=0$, be two equations, containing two unknown quantities, x and y, of which we know approximate and co-ordinate values a and b respectively, so that $x=a+h$, $y=b+k$, when h and k are both very small; then $\phi(a+h, b+k)$ may be expanded in the form $\phi(a, b)+Ah+Bk=0$, neglecting higher powers of h, k, and their rectangles, as compared with their first powers; similarly, we shall have $F(a, b)+A'h+B'k=0$, from which simple equations the first approximations to the correct values of h and k are readily deduced.

Example. To find the imaginary roots of the equation $x^3-2x-5=0$.

Let α, β be the imaginary roots, of which let the sum be s and the product p; let α, β be each put for x in the equation, and by summing the results we find, $s^3-3sp-2s-10=0$, observing that $\alpha^3+\beta^3=s^3-3sp$. Again, if we subtract the result of one substitution from the other and divide the remainder by $\alpha-\beta$, we find $\alpha^2+\alpha\beta+\beta^2-2=0$, or $s^2-p-2=0$, we must now seek approximate and co-ordinate values of s and p, as S, P, and then putting $s=S+h$ $p=P+k$, when h and k are supposed very small, we shall have

(1) . . . $(S^3-3SP-2S-10)+h(3S^2-3P-2)-3Sk=0.$

(2) $(S^2-P-2)+2Sh-k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . =0.$

(3) Hence $h=-2 \cdot \dfrac{S^3-2S+5}{3S^2+3P+2}$; $k=S^2-P-2+2Sh$.

The cotemporary values of x and y render $S^2-p-2=0$ and their approximate values must render S^2-P-2 a small quantity; if we put $S+h$ for S, $P+k$ for P, we shall obtain second corrections h', k', and putting $S+h+h'$ $P+k+k'$ for S and P, third corrections will be found, &c.

Form now a recurring series, of which the first three terms are 1, 2, 4, and the scale of constants are the coefficients in the given equation, namely 0, 2, 5;

1, 2, 4, 9, 18, 38, 81, 166, 352, 737, 1534, 3234, 6753, 14138,

and taking the successive quotients we find them convergent, though sometimes above and sometimes below the value to which they converge; if we take only the last 3 terms we have

$$\frac{6753}{3234}=2.093. \qquad \frac{14138}{6753}=2.099.$$

The real root being, therefore, 2.09 nearly, we must have $S=-2.09$, and dividing 5 the product of all the roots by 2.09, the quotient is 2.39 . . . We may therefore take $P=2.40$.

Hence $S^3-2S+5=.0506...$ $\qquad S^2-P-2=-.0319$

$3S^2+3P+2=22.3043.$

Hence $h=-.90045$ $\qquad k=-,0319+4,18.h=-,0131.$

The corrected values of S and P are thus, $S=-2,0945...P=2,3869,$ the first of which is correct to the last figure, and the second is too small only by .0001 ... And a second approximation is readily effected by writing these values for S and P in the formula above given. The required imaginary roots are therefore $-1,0472. \; . \; .\pm 1,1362. \, . \, .\sqrt{-1}$.

Example 2. To find all the roots of the equation $x^4+x+10=0$ (no possible roots), we must form a primitive recurring series, of which the scale of constants is 0, 0, −1, −10, and the first four terms may be any assumed: we shall take those corresponding to the sums of the first, second, third, and fourth powers of the roots, viz. 0, 0, −3, −40.

Primitive series, 0, 0, −3, −40, 0, 3, 70, 400, −3, −100, −1100, −3997, 130, 2100, 14997, 39840, −3400.

From the product of the extremes belonging to any four consecutive terms of the primitive series subtract the product of the means, and thus another recurring series is generated.

1st derived series 0, −120, −9, −2800, −210, −28009, −5800, −440300, −98009, −4409700. . .in which the ratios of the consecutive terms, alternately greater and less than unity, have but little convergence to any definite value; therefore it will be of advantage to form a second primitive series, of which the first four terms shall be nearly *proportional* to the four last-computed terms of the preceding and lower in numerical value, as, for example, when divided by 700 we have thus

Second primitive series, 3, 21, 57, −5, −51, −267, −565, 101, 777, from which the series derived in the manner above mentioned, changing all the signs, will be

1212, 786, 15475, 10792, 156006, 150394.

In the former derived series the limits of the product of the *first* pair of impossible roots which are found by taking the quotients of the successive terms were $\frac{2}{9}$ and 45 nearly, and in the latter one they are 1 and $14\frac{2}{5}$ nearly, which are yet too remote; another primitive series is therefore to be formed, taking −5, −11, 2, 15 for the first terms which arise by dividing the four last terms of the above by 51 to obtain low results.

Third prim. series,

−5, −11, 2, 15, 61, 108, −35, −211, −718, −1045, 561, 2828

and the three last terms of the derived, with signs taken positively, are

114923, 868681, 1444259,

when the successive quotients are nearly $7\frac{1}{2}$ and $1\frac{2}{3}$.

Divide the four last terms of this primitive series by 51 and commence a fourth with the quotients.

Fourth prim. series,

−14, −20, 11, 55, 160, 189, −165, −710, −1789, −1625, 2360,

from which we obtain for the three last terms of the derived recurring series

455271, 1002065, 4582725.

The successive quotients are now $2\frac{1}{4}$, $4\frac{1}{2}$ nearly; dividing by 71 we get similarly

−10, −25, −23, 33, 125, 273, 197, −455, −1523, −2927, −1515.

Last derived terms, with signs changed, are

326144, 1269584, 3768496,

the limits now obtained are 2, 97. . . and 3, 8. . .

Finally form a primitive series, dividing the four last terms of the above by -152,

$$3, 10, 19, 10, -40, -119, -200, -60, 519, 1390, 2060, 81,$$

the three last terms of the derived are 246860, 845010, 2821361, from whence we find 3·34 and 3·42 very nearly as the limits of the product of the first pair of imaginary roots, and since the product of all the roots is 10, that of the second pair is between 2.92 and 3.

Again, a second derived series may be formed by taking from the product of the extremes of 3 terms the square of the mean, and changing the sign of the remainder, the two last of which corresponding to the two last above obtained are $(519)^2+60\times1390$ and $(1390)^2-519\times2060$, or 352761 and 861960, therefore the limits for the sum of the first pair of impossible roots are $\frac{845010}{352761}$ and $\frac{2821361}{861960}$ or 2.4 and 3.2 nearly, the same quantities taken negatively being the limits for the sum of the second pair.

The proper limits being thus obtained, let s represent the sum and p the product of the first pair of roots α, β, then we have

$$\alpha^4+\alpha+10=0 \qquad \beta^4+\beta+10=0$$

But $\alpha^4+\beta^4=s^4-4s^2p+2p^2$; $\frac{\alpha^4-\beta^4}{\alpha-\beta}=(\alpha^3+\beta^3)+\alpha\beta(\alpha+\beta)=s^3-2sp$.

Hence
$$s^4-4s^2p+2p^2+s+20=0$$
$$s^3-2s\,p+1 \qquad =0$$

and to obtain a first approximation, let

$$S=2.5,\quad P=3.4,\quad s=S+h,\quad p=P+k.$$

Hence $(S^4-4S^2P+2P^2+S+20)+(4S^3-8SP+1)h-4(S^2-P)k=0.$

$$(S^3-2SP+1)+(3S^2-2P)h-2Sk\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.=0,$$

or
$$,0625-4,5h-11.4k=0$$
$$-,375+11,95h-5k..=0$$

from whence we find $h=,0028$ $k=,0043$, and for a first approximation $S=2.5028$ $P=3.4043$ and one half the difference of these two roots $=\sqrt{\left(\frac{S^2}{4}-P\right)}=1,348\ldots\sqrt{-1}$; hence the first pair of impossible roots are $1,251\ldots\pm1,348\ldots\sqrt{-1}$.

For the second pair the sum is -2.5028, and the product $=P'=\frac{10}{P}=3,231$, therefore their semi-difference $=\sqrt{\left(\frac{S^2}{4}-P'\right)}=1,282\ldots$ therefore the second pair are $-1,251\ldots\pm1,282\ldots\sqrt{-1}$.

This example from the near equality of the moduli of the impossible roots (that is the square root of the product of each pair) was unfavourable for the application of the method of recurring series; but as this method may be said to be the only direct one known for obtaining a first notion of the magnitudes of the real and imaginary parts

of the roots of equations, we have, therefore, developed it at length, that it may be seen how we are to improve the first arbitrary terms of the primitive recurring series, namely by commencing other series, the first terms of which will be small and nearly proportional to the last terms of the previous recurring series; we may add that the research of the impossible roots of equations has been generally overlooked in modern treatises of algebra, although it is requisite for the integration of rational fractions.

(86.) Method of Continued Fractions.

Let $\phi(x)=0$ represent a proposed algebraical equation, which we suppose to have some real roots, and in this method the research is confined to the positive roots, and the negative roots may be found in the same manner by making $x=-y$ for then all the negative values of x correspond to positive values of y.

Divide $\phi(x)$ by the derived function $\phi'(x)=V_1$, and change the sign of the remainder; let this quantity V_2 be made a divisor in the same way for $\phi'(x)$ and the next remainder with a changed sign V_3 a divisor for V_2, and so on until we arrive at a constant V_n; then when x is put $=0$ in the series of functions $\phi(x)$, V_1, V_2, $V_3 \ldots\ldots V_n$, they are reduced to their last or absolute terms, and when $x=+\infty$ their signs are the same as those of their first terms; observe how many alternations of signs there are (from $+$ to $-$) more in their last than in their first terms; we shall thus know how many positive roots exist.

Substitute in the same functions 1, 10, 100, 1000. . . . successively for x until the results have as few alternations of signs as the first terms of the functions; then observing how many alternations are lost from $x=1$ to 10, from $x=10$ to 100, &c., we shall know how many real roots there are in the same limits.

If roots are between 10 and 100 then put 20, 30, 40. . . . 90 for x, and if, for example, we found them yet between 30 and 40, then put 31, 32, 33, ...39 for x, and thus it is plain we shall separate all the roots of which the difference is greater than unity, and moreover we shall become acquainted with the integer part of the root or roots; in other words the integer which is next less than each root.

Let p be an integer thus found which may belong to only one, or be common to several roots, according as the functions $\phi(x)$, V_1, $V_2 \ldots . V_n$, lose only one or several alternations of signs from $x=p$ to $x=p+1$. To the remaining roots would correspond other integer parts, p', p'', &c., to which should be applied a similar process to that about to be described.

Make $x=p+\dfrac{1}{y}$, the transformed equation is

$$\phi(p)\cdot y^n+\phi'(p)\cdot y^{n-1}+\frac{\phi''(p)}{1.2}\cdot y^{n-2}+\ldots.\frac{\phi'''^{(n)}(p)}{1.2\ldots n}=0$$

where n is used to denote the dimensions of $\phi(x)$.

Our object being to approximate to the value or values of y, which are between p and $p+1$, we must take values of y in the transformed equation which are greater than unity, and positive; the number of these we already know, being the same as the number of values of x in the above-mentioned interval. We can now find the integer part of the value or values of y in the same manner we had employed for x;

but it should be observed, that when there is only one value of x between p and $p+1$, there can be only one positive value of y greater than unity, it will be therefore sufficient to substitute the natural numbers, 1, 2, 3, &c., for y in such case, until the transformed function changes signs; we shall thus know the integers q, $q+1$, which limit y.

We have now merely to repeat this process, making $y=q+\frac{1}{z}$, $z=r+\frac{1}{u}$, &c., and however near two roots may be, we shall ultimately separate them by this process.

Such is the method of continued fractions given by Lagrange, but the certainty of its application we see depends on Sturm's theorem.

After the roots are separated, we may, by similar transformations, converge to one of them, the continued fraction or value of x being

$$p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s}+\text{\&c.}}}$$

and it is easily seen that the converging fractions which are deduced are alternately greater and less than the true value; as such transformations are, however, in many cases tedious and laborious, it is practically better to use Newton's method of approximation, after the separation of the roots.

(87.) To find the imaginary roots, Lagrange recommends to form the equation to the squares of the differences of the roots, so that if $\alpha+\beta\sqrt{-1}$ and $\alpha+\beta\sqrt{-1}$ were roots of the proposed, $-4\beta^2$ would be a real root of the transformed and is essentially negative; but besides the great labour necessary in general to form this equation, and the high dimensions $\frac{n(n-1)}{2}$ to which it rises, the labour of approximating to β by means of this equation, and then of approximating to α by means of the equation $\phi(\alpha+\beta\sqrt{-1})-\phi(\alpha-\beta\sqrt{-1})=0$; (which, generally speaking, render the method all but impracticable); besides all these objections, there are several causes to produce an uncertainty in the determinations, even when all the labour is surmounted. For instance, when two pairs of impossible roots $\alpha+\beta\sqrt{-1}$, $\alpha-\beta\sqrt{-1}$, $\alpha+\gamma\sqrt{-1}$, $\alpha-\gamma\sqrt{-1}$, have the possible part α common to all, then $-(\beta-\gamma)^2$, $-(\beta+\gamma)^2$, will be negative roots of the transformed, as well as $-4\beta^2$, $-4\gamma^2$, though they correspond to no conjugate pair of impossible roots of the proposed. We, therefore, in seeking such roots, decidedly recommend the method of recurring series, as before explained, used with a due regard to the diminution of labour by proper assumptions; this being the only other method existing for the purpose in the present state of analysis, putting out of consideration the purely tentative method of forming a table to double entry, in which a series of values would be assigned to β, corresponding to every assigned value of α.

(88.) Consider now the fractions converging to the ultimate value of the continued fraction,

$$p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s+\ \&c.}}}$$

they are p, $p+\frac{1}{q}$, $p+\cfrac{1}{q+\frac{1}{r}}$, &c., which may be reduced to simple fractions, capable of being successively formed, each from two preceding fractions, in an easy manner.

The $(n+1)$th converging fraction being represented by $\frac{\alpha_n}{\beta_n}$, we have manifestly

$$\frac{\alpha_0}{\beta_0}=\frac{p}{1} \quad \text{or} \quad \alpha_0=p \quad \beta_0=1$$

$$\frac{\alpha_1}{\beta_1}=\frac{pq+1}{q} \quad \text{or} \quad \alpha_1=q\alpha_0+1 \qquad \beta_1=q\beta_0 ;$$

in this formula write $q+\frac{1}{r}$ for q, and multiply numerator and denominator by r, the result will be the next converging fraction.

Hence $$\frac{\alpha_2}{\beta_2}=\frac{\left(q+\frac{1}{r}\right)\alpha_0+1}{\left(q+\frac{1}{r}\right)\beta_0}=\frac{r\,(q\,\alpha_0+1)+\alpha_0}{r\,.\,q\beta_0+\beta_0}=\frac{r\,\alpha_1+\alpha_0}{r\,\beta_1+\beta_0}$$

$$\text{or}\ \alpha_2=r\,\alpha_1+\alpha_0 \qquad \beta_2=r\beta_1+\beta_0$$

In the same manner by writing $r+\frac{1}{s}$ for r we obtain

$$\frac{\alpha_3}{\beta_3}=\frac{\left(r+\frac{1}{s}\right)\alpha_1+\alpha_0}{\left(r+\frac{1}{s}\right)\beta_1+\beta_0}=\frac{s\,(r\,\alpha_1+\alpha_0)+\alpha_1}{s\,(r\,\beta_1+\beta_0)+\beta_1}=\frac{s\,\alpha_2+\alpha_1}{s\,\beta_2+\beta_1};$$

that is $\alpha_3=s\,\alpha_2+\alpha_1 \qquad \beta_3=s\beta_2+\beta_1;$

and in general if u be the nth and v the $(n+1$th) denominator of the simple fractions, and if $\frac{\alpha_n}{\beta_n}=\frac{u\alpha_{n-1}+\alpha_{n-2}}{u\beta_{n-1}+\beta_{n-2}}$, we see that α_{n-1}, α_{n-2}, β_{n-1}, β_{n-2}, depend only on the denominators which precede v, and should not be altered when, to obtain the next converging fraction, we write $u+\frac{1}{v}$ for u, therefore

$$\frac{\alpha_{n+1}}{\beta_{n+1}}=\frac{\left(u+\frac{1}{v}\right)\alpha_{n-1}+\alpha_{n-2}}{\left(u+\frac{1}{v}\right)\beta_{n-1}+\beta_{n-2}}$$

$$=\frac{v(u\,\alpha_{n-1}+\alpha_{n-2})+\alpha_{n-1}}{v(u\,\beta_{n-1}+\beta_{n-2})+\beta_{n-1}}$$

$$=\frac{v\,\alpha_n+\alpha_{n-1}}{v\,\beta_n+\beta_{n-1}}$$

and therefore $\alpha_{n+1}=v.\alpha_n+\alpha_{n-1}$ $\beta_{n+1}=v\beta_n+\beta_{n-1}$, whence we see the generality of this law of formation which may be thus announced.

Calling the denominators q, r, s, &c., the partial quotients, then, to form the converging fraction when we stop at any given partial quotient, knowing the two preceding converging fractions, multiply the numerator which occupies one place before that sought by the partial quotient for the latter, and to this product add the numerator two places before.

Multiply the denominator of the place before by the same partial quotient, and to the product add the denominator two places before.

We shall thus have the new numerator and the new denominator.

Example. Let the continued fraction be

$$\cfrac{1}{3+\cfrac{1}{3,\ \&c.}}$$

The first two converging fractions are $\frac{1}{3}$ and $\frac{3}{10}$, from which the others can be formed as follows by the rule given,

$$\frac{1}{3},\ \frac{3}{10},\ \frac{10}{33},\ \frac{33}{109},\ \frac{109}{360},\ \frac{360}{1189},\ \&c.$$

The exact value to which the fractions in this example converge is incommensurate, for if we represent it by x, then the actual denominator taken after the first dividing line of the continued fraction is $3+x$, and therefore,

$$x=\frac{1}{3+x} \qquad \text{or} \qquad x^2+3x=1$$

$$x=-\frac{3}{2}\pm\sqrt{\frac{13}{4}};$$

but since the fraction is clearly positive, we must take only the upper sign, and therefore its correct value is $\frac{1}{2}\{\sqrt{(13}-3\}$

(89.) Every continued fraction, of which the partial denominators circulate, is a root of a quadratic equation;

For let $x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \&c.}}}$

and suppose the partial denominator to circulate after n places, that is, let $a_{n+1} = a_1 \quad a_{n+2} = a_2$, &c. . . . $a_{2n+1} = a_1$, &c.

Let $\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \frac{\alpha_3}{\beta_3}$, &c., be the converging fractions, then since

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots\cdots + \cfrac{1}{a_n + x}}}};$$

if we consider $a_n + x$ as one partial quotient, the corresponding converging fraction ought to be the same as x, and by the rule this is

$$\frac{(a_n + x)\,\alpha_{n-1} + \alpha_{n-2}}{(a_n + x)\,\beta_{n-1} + \beta_{n-2}} = x,$$

from which it is obvious that x is a quadratic surd; the converse of this proposition, namely, that every quadratic surd generates a continued fraction with recurring periods, we shall shortly find to be also true.

(90.) Recurring now to the general laws of forming the numerator and denominator of the converging fractions, we can eliminate the partial quotient v, and thus arrive at a relation independent of such quotient; we have $\alpha_{n+1} = v.\alpha_n + \alpha_{n-1}$

$$\beta_{n+1} = v.\beta_n + \beta_{n-1};$$

therefore $\beta_n\alpha_{n+1} - \alpha_n\beta_{n+1} = \beta_n\alpha_{n-1} - \alpha_n\beta_{n-1}$;

from which we learn that, when n is increased to $n+1$, the quantity $\beta_n\alpha_{n-1} - \alpha_n\beta_{n-1}$ retains the same magnitude, but changes its sign. Now, when n is unity, its value is $q.p - (q\alpha_0 + 1) = -1$; therefore, when n is 2, the value is $+1$; when n is 3, it is -1, and generally $\beta_n\alpha_{n-1} - \alpha_n\beta_{n-1} = (-1)^n$.

Thus, in the numerical example above given, we have the two consecutive fractions $\frac{109}{360}$, and $\frac{360}{1189}$, accordingly we find

$$1189 \times 109 - (360)^2 = 1.$$

From this it follows that every numerator in the series of converging fractions is prime to the corresponding denominator and to the preceding numerator, for if α_{n-1}, α_n had any common measure, or α_n, β_n a common measure, the same must measure $(-1)^n$, which is impossible.

It also follows that the difference of two consecutive fractions is alternatively positive and negative, and is a fraction, of which the nume-

rator is 1, and the denominator the product of two consecutive denominators.

Thus $\frac{\alpha_{n-1}}{\beta_{n-1}} - \frac{\alpha_n}{\beta_n} = \frac{\alpha_{n-1}\beta_n - \alpha_n \beta_{n-1}}{\beta_{n-1}\beta_n} = \frac{(-1)^n}{\beta_{n-1}\beta_n}$, which, abstracting from its sign, is less than $\left(\frac{1}{\beta_{n-1}}\right)^2$; and since, moreover, the converging fractions are alternately greater and less than the correct value of the continued fraction, the error is less than unity divided by the square of the denominator of the fraction so obtained. Thus, the difference between $\frac{109}{306}$ and $\frac{1}{2}\{\sqrt{13} - 3\}$ is less than the difference between $\frac{109}{360}$ and $\frac{360}{1189}$; that is, less than $\frac{1}{428040}$. Thus every fraction approaches much more nearly to the true value than the preceding.

Suppose now $\frac{\alpha_n}{\beta_n}$ to be below the true value, and $\frac{P}{Q}$ any other fraction also below the true value, but nearer to it than $\frac{\alpha_n}{\beta_n}$, then $\frac{P}{Q} < \frac{\alpha_{n+1}}{\beta_{n+1}}$, since the converging fractions are alternately above and below the true value: hence $\frac{P}{Q} - \frac{\alpha_n}{\beta_n}$ is positive, and less than $\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}$, or $\frac{1}{\beta_n \beta_{n+1}}$; therefore $P\beta_n - Q\alpha_n$ is a positive *integer*, but less than $\frac{Q}{\beta_{n+1}}$. Q therefore cannot be less than β_{n+1}; in other words, we can interpolate no fraction of a denominator inferior to β_{n+1} which shall be nearer the true value than $\frac{\alpha_n}{\beta_n}$, and *à fortiori* than $\frac{\alpha_{n+1}}{\beta_{n+1}}$.

(91.) We proceed to illustrate by examples the theory here explained, with the facility given to its application by Sturm's theorem.

Example. Given $x^3 - 7x^2 + 17x - 13 = 0$.

$$
\begin{array}{l}
3x^2 - 14x + 17)\,3x^3 - 21x^2 + 51x - 39\,(x \\
\qquad\qquad\qquad\; 3x^3 - 14x^2 + 17x \\
\qquad\qquad\qquad\quad - 7x^2 + 34x - 39 \\
\qquad\qquad\qquad\quad -21x^2 + 102x - 117\,(-7 \\
\qquad\qquad\qquad\quad -21x^2 + 98x - 119 \\
\qquad\qquad\qquad\qquad\qquad 4x + 2 \\
\qquad\qquad\qquad\qquad -2x - 1)\,6x^2 - 28x + 34\,(-3x \\
\qquad\qquad\qquad\qquad\qquad\quad 6x^2 + 3x \\
\qquad\qquad\qquad\qquad\qquad\qquad -31x + 34 \\
\qquad\qquad\qquad\qquad\qquad\qquad -62x + 68\,(31 \\
\qquad\qquad\qquad\qquad\qquad\qquad -62x - 31 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad +99 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -99
\end{array}
$$

The equation has but one real root, which lies between 1 and 2.

Put $x = 1 + \frac{1}{y}$ $\phi(x) = x^3 - 7x^2 + 17x - 13$;

therefore $\phi(1) = -2$ $\phi'(1) = 6$ $\frac{\phi''(1)}{1.2} = -4$ $\frac{\phi'''(1)}{1.2.3} = 1.$

The transformed is $2y^3 - 6y^2 + 4y - 1 = 0$.

There is a root (the only one) between 2 and 3.

Put $y = 2 + \frac{1}{z}$ $\phi_1(y) = 2y^3 - 6y^2 + 4y - 1$

$$\phi_1(2) = -1 \quad \phi_1'(2) = 4 \quad \frac{\phi_1''(2)}{1.2} = 6 \quad \frac{\phi_1'''(2)}{1.2.3} = 2;$$

therefore $z^3 - 4z^2 - 6z - 2 = 0$.

The root is between 5 and 6.

Put $z = 5 + \frac{1}{u}$ $\phi_2(z) = z^3 - 4z^2 - 6z - 2$

$$\phi_2(5) = -7 \quad \phi_2'(5) = 29 \quad \frac{\phi_2''(5)}{1.2} = 11 \quad \frac{\phi_2'''(5)}{1.2.3} = 1;$$

therefore $7u^3 - 29u^2 - 11u - 1 = 0$.

The value of u is between 4 and 5.

These transformations may be easily continued. Hence

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{4 + \&c.}}}$$

The converging fractions are $\frac{1}{1}, \frac{3}{2}, \frac{16}{11}, \frac{67}{46} \ldots\ldots$

Example 2. Given $8x^3 - 6x + 1 = 0 = \phi(x)$ $\phi'(x) = 6(4x^2 - 1)$.

$$\begin{array}{l}
4x^2-1)\,8x^3-6x+1\,(2x \\
\quad\quad\;\; \underline{8x^3-2x} \\
\quad\quad\quad\;\; -4x+1 \\
\quad\quad\quad\quad 4x-1)\,4x^2-1\,(x \\
\quad\quad\quad\quad\quad\quad\; \underline{4x^2-x} \\
\quad\quad\quad\quad\quad\quad\quad x-1 \\
\quad\quad\quad\quad\quad\quad\; 4x-4\,(1 \\
\quad\quad\quad\quad\quad\quad\; \underline{4x-1} \\
\quad\quad\quad\quad\quad\quad\quad\; -3 \\
\quad\quad\quad\quad\quad\quad\quad\; +3.
\end{array}$$

The series of functions in this case is

$$8x^3 - 6x + 1,\ 4x^2 - 1,\ 4x - 1,\ +3.$$

The equation has therefore no impossible root.

Put $x = 0$. The signs are $+, -, -, +$; and when $x = +\infty$, they are $+ + + +$; therefore the equation has two positive roots.

Put $x=1$. The signs are $+\ +\ +\ +$.

The two positive roots lie therefore between 0 and 1.

Put $x=0+\dfrac{1}{y}$; therefore $y^3-6y^2+8=0=\phi_1(y)$ $\phi_1'(y)=3(y^2-4y)$.

$$y^2-4y)y^3-6y^2+8(y-2$$
$$\underline{y^3-4y^2}$$
$$-2y^2+8$$
$$\underline{-2y^2+8y}$$
$$-8y+8$$
$$y-1)y^2-4y(y-3$$
$$\underline{y^2-\ y}$$
$$-3y$$
$$\underline{-3y+3}$$
$$-3$$
$$+3.$$

The series of functions for determining the limits of y is therefore

$$y^3-6y^2+8,\ y\,(y-4),\ y-1,\ +3.$$

Put $y=0$; resulting signs are	$+$	*	$-$	$+$
$y=1$ - - - -	$+$	$-$	*	$+$
$y=2$ - - - -	$-$	$-$	$+$	$+$
$y=3$ - - - -	$-$	$-$	$+$	$+$
$y=4$ - - - -	$-$	*	$+$	$+$
$y=5$ - - - -	$-$	$+$	$+$	$+$
$y=6$ - - - -	$+$	$+$	$+$	$+$

Hence there is a root between 1 and 2, and another between 5 and 6.

Put therefore $y=1+\dfrac{1}{z}=5+\dfrac{1}{z'}$

$$\phi_1(y)=y^3-6y^2+8\,;\ \phi_1'(y)=3(y^2-4y)\,;\ \frac{\phi_1''(y)}{2}=3(y-2)\frac{\phi_1'''(y)}{2.3}=1$$

$$\phi_1(1)=3;\qquad \phi_1'(1)=-9;\qquad \frac{\phi_1''(1)}{2}=-3$$

$$\phi_1(5)=-17;\qquad \phi_1'(5)=15;\qquad \frac{\phi''(5)}{2}=9;$$

therefore
$$2z^3-9z^2-3z+1=0$$
$$17z'^3-15z'^2-9z'-1=0,$$

when z and z' have each only one value greater than unity.

By observing when the preceding functions of z and z' change sign, we find z is between 3 and 4, z' between 1 and 2.

Put therefore $z=3+\dfrac{1}{u}$ $z'=1+\dfrac{1}{u'}$;

the transformations and continuation of the process may be then easily performed, as in the preceding example: whence the two positive roots

of x are $0+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{u}}}$, and $0+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{u'}}}$.

Again, in the primitive series of functions, which define the limits of the roots of the proposed equation, viz.;

$$8x^3-6x+1,\ 4x^2-1,\ 4x-1,\ +3.$$

Put $x=0$, the signs are $+\ -\ -\ +$

$x=-1$ - - - $-\ +\ -\ +$;

therefore there is a negative root between 0 and -1.

To find this root, make $x=0-\dfrac{1}{y'}$;

hence $$y'^3+6y'^2-8=0:$$

y' must have one positive value greater than unity: it is easily found to be between 1 and 2. Put $y'=1+\dfrac{1}{z''}$, the transformed equation is

$z''^3-15z''^2-9z''-1=0$.

Hence z'' lies between 15 and 16, or $z''=15+\dfrac{1}{u''}$.

The required negative root is therefore $0-\cfrac{1}{1+\cfrac{1}{15+\cfrac{1}{u'}}}$;

from which we see that the negative roots may be approximated to in the same manner as the positive.

We shall next take for our example one which has already been solved by a different method, that the accuracy of the approximations may be compared.

Example 3. $x^3-2x-5=0$.

By the method so often used before for finding the limits of the roots of equations, we find that x in this case lies between 2 and 3; we must therefore make $x=2+\dfrac{1}{y}$; and, multiplying the transformed equation by y^3, and changing all the signs, we have

$$y^3-10y^2-6y-1=0.$$

Substitute for y the natural positive integers until this function of y changes its sign from negative to positive, the equation, as we have shown, having only a single positive root greater than unity; and, since the greatest negative coefficient of an equation increased by unity is a superior limit to the roots, if we commence the substitutions, putting for y 11 and then 10, 9, &c., the result has contrary signs for the first two; therefore y is between 10 and 11.

The next transformation arises by putting $y=10+\dfrac{1}{z}$, whence we find

$$61z^3-94z^2-20z-1=0;$$

and, since the greatest negative coefficient when the equation has been divided by 61 is $\frac{94}{61}$, 2 is a superior limit; substitute, therefore, 0, 1, 2 for z, and the function changes sign, the results being -1, -54, $+71$, &c.; hence z is between 1 and 2.

Next make $z=1+\frac{1}{u}$, and we find the transformed to be

$$54u^3+25u^2-89u-61=0;$$

hence $\frac{89}{54}+1$ is a superior limit to the positive root of the equation; and, upon substituting, we find that u is between 1 and 2.

Similarly we put $u=1+\frac{1}{s}$, and we find s between 2 and 3. $s=1+\frac{1}{t}$, then t is between 1 and 2. $t=1+\frac{1}{r}$, and r is between 3 and 4. $r=3+\frac{1}{k}$, k is between 1 and 2, &c.

Hence $x=2+\cfrac{1}{10+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{12+\&c.}}}}}}}}}$;

from whence we form the converging fractions

$$\frac{2}{1},\ \frac{21}{10},\ \frac{23}{11},\ \frac{44}{21},\ \frac{111}{53},\ \frac{155}{74},\ \frac{576}{275},\ \frac{731}{549},\ \frac{1307}{6241},\ \frac{16415}{7837},$$

which are alternately greater and less than the true root, the error of the last being less than $\frac{1}{(7837)^2}$, or, converted into a decimal, less than .0000000163. Now this fraction $\frac{16415}{7837}$ is the same as 2.0945514865, and by Newton's method of approximation we have already found the true root to be 2.0945514815, the error being only equal to .0000000051.

The other two roots of the equation are imaginary; but, as the dimensions of the proposed are in this case low, they may be readily enough found by taking the equation of which the roots are the squares of the differences of the roots of the given equation—the method used by Waring and Lagrange.

Let $a+\beta\sqrt{-1}$ and $a-\beta\sqrt{-1}$ be the imaginary roots, the difference of which is $2\beta\sqrt{-1}$, and the square of the difference is $-4\beta^2$. Now, if $-X$ be the square of the difference of any two roots of the proposed,

the equation for determining X (the method for forming which we have given at the beginning of this work) is

$$X^3+12X^2+36X-643=0=\phi(X) \text{ suppose};$$

$$3(X^2+8X+12)=\phi' X;$$

$$\begin{array}{l}
X^2+8X+12\,)\,X^3+12X^2+36X-643\,(\,X+4 \\
\qquad\qquad\quad \underline{X^3+\ 8X^2+12X} \\
\qquad\qquad\qquad\quad 4X^2+24X-643 \\
\qquad\qquad\qquad\quad \underline{4X^2+32X+\ 48} \\
\qquad\qquad\qquad\qquad -\ 8X-691 \\
\qquad\qquad\qquad\qquad\quad 8X+691\,)\,8X^2+\ 64X+\ 96\,(\,X \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{8X^2+\ 691X} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -\ 627X+\ 96 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -5016X+768\,(\,-627 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{-5016X-627\times 691} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{remainder positive.}
\end{array}$$

The functions giving the limits in this case are

$X^3+12X^2+36X-643$, $X^2+8X+12$, $8X+691$ and a negative constant.

$$\text{Put } X=5;\ \text{the signs are}\ -\ +\ +\ -$$

$$X=6\ .\ .\ .\ .\ +\ +\ +\ -;$$

therefore X is comprised between the numbers 5 and 6.

Make now $X=5+\dfrac{1}{Y}$, and, changing the signs to render the first term of the transformed positive, we have

$$38Y^3-231Y^2-27Y-1=0;$$

a superior limit to the root of which is $\dfrac{231}{38}+1$, or 8. Substitute 8, 7, 6, &c. for Y, and we find a change of sign in the function in the two latter; therefore Y is between 6 and 7.

Hence $Y=6+\dfrac{1}{Z}$; the transformed equation is

$$271Z^3-1305Z^2-453Z-38=0;$$

the superior limit to which is $\dfrac{1305}{271}+1$, or 6; and by the same process repeated we find $z=5+\dfrac{1}{U}$, $U=6+\dfrac{1}{V}$, &c.;

whence
$$X=5+\cfrac{1}{6+\cfrac{1}{5+\cfrac{1}{6+\&c.;}}}$$

the converging fractions to which are

$$\frac{5}{1}, \frac{31}{6}, \frac{160}{31}, \frac{991}{192}, \&c.$$

We thus find $X=5.161458\ldots$; the error being less than $.000002$.

Hence $2\beta=\sqrt{X}=2.27188\ldots$ $\beta=1.13594\ldots$

Now, if we put $\alpha+\beta\sqrt{-1}$ for x in the proposed equation, and equate the possible and impossible parts separately to zero, we have

$$\alpha^3-(3\beta^2+2)\alpha-5=0$$
$$3\alpha^2-\beta^2-2=0.$$

Multiply the former by 3 and the latter by α, and take the difference of the product;

$$\alpha=-\frac{15}{8\beta^2+4}=-\frac{15}{2(X+2)},$$

whence the sought imaginary roots are

$$-1.04727\ldots \quad \pm 1.13594\ldots \quad \sqrt{-1};$$

to which, if we desire greater accuracy, we may now apply Simpson's extension of Newton's method of approximation.

Example 4, for practice.

$$x^3-7x-7=0.$$

This equation has two positive roots and one negative; they are

$$x=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{4+\&c.}}}$$

$$x=1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+\&c.}}}$$

$$x=-3-\cfrac{1}{20+\cfrac{1}{3+\&c.}}$$

It may be observed that all the successive transformations required in the application of this method arise from substitutions of the form $x=\frac{a+bz}{a'+b'z}$, the quantities a, b, a', b' being of course different in the different transformations.

When a proper fraction $\frac{p}{q}$ is reduced to a continued fraction, the number of quotients or partial fractions generated depends on q, but q being given it varies with p, we can find a limit to that varying number in the following manner.

Supposing p to be less than q, divide q by p, let the quotient be a_1 and the remainder p_1, hence $\frac{p}{q} = \cfrac{1}{a_1 + \frac{p_1}{p}}$.

Again let p be divided by p_1, let a_2 be the second quotient, and p_2 the second remainder, then $\frac{p}{q} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{p_2}{p_1}}}$.

This quantity p_2, or second remainder, is necessarily less than the half of p, for if p_1 be one-half or less than one-half of p, it is clear that p_2 which is less than p_1, is less than $\frac{p}{2}$, and if p_1 be greater than the half of p, then the quotient a_2 is necessarily unity, and $p_2 = p - p_1$ is yet less than $\frac{p}{2}$.

For the same reason when $\frac{p_2}{p_1}$ is reduced to the form $\cfrac{1}{a_3 + \cfrac{1}{a_4 + \frac{p_4}{p_3}}}$

we must have $p_4 < \frac{p_2}{2} < \frac{p}{2^2}$ similarly $p_6 < \frac{p}{2^3}$, and generally $p_{2n} < \frac{p}{2^n}$.

If, therefore, 2^n be the nearest power of 2 less than p, p_{2n} must be 1 or 0, and the number of partial fractions cannot exceed $2n$.

Since p may be reduced by division if it exceeds q, it follows generally, that taking 2^m for the power of 2, which is nearest below q, the number of partial fractions in the continued fraction which represents $\frac{p}{q}$ cannot exceed $2m$.

Example. $\frac{7}{11} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{3}}}}$.

Since $2^2 = 4$ is the nearest power of 2 below 7, the number of partial quotients cannot exceed double the index, which is 4; but is in this case exactly equal to it.

(91.) *Inversion of continued Fractions.*

It will be convenient to adopt a notation to express more briefly a *general* continued fraction, as $\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots + \frac{1}{a_n}}}}$.

It will be only necessary to introduce the two extreme quotients a_1, a_n in the order in which they stand; thus this continued fraction will be understood from the expression $(a_1 \ldots . a_n)$.

Let $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$, $\frac{p_3}{q_3} \ldots \ldots \frac{p_n}{q_n}$ be the converging fractions, then by this notation we shall have

$$(a_1) = \frac{p_1}{q_1} \quad (a_1, a_2) = \frac{p_2}{q_2} \quad (a_1 . . a_3) = \frac{p_3}{q_3} \ldots \ldots (a_1 \ldots . a_n) = \frac{p_n}{q^n} .$$

Theorem. If $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$ be the two final converging fractions to a given continued fraction $(a_1 \ldots \ldots a_n)$, the value of the inverted fraction $(a_n \ldots . a_1)$ will be $\frac{q_{n-1}}{q_n}$.

For we know by the formation of the converging fractions that

$$q_n = a_n q_{n-1} + q_{n-2},$$

$$\text{therefore,} \quad \frac{q_n}{q_{n-1}} = a_n + \frac{q_{n-2}}{q_{n-1}},$$

$$\text{whence} \quad \frac{q_{n-1}}{q_n} = \cfrac{1}{a_n + \cfrac{q_{n-2}}{q_{n-1}}}$$

changing successivly n into $n-1$, $n-2$, &c., this formula will give

$$\frac{q_{n-2}}{q_{n-1}} = \cfrac{1}{a_{n-1} + \cfrac{q_{n-3}}{q_{n-2}}}$$

$$\frac{q_{n-3}}{q_{n-2}} = \cfrac{1}{a_{n-2} + \cfrac{q_{n-4}}{q_{n-3}}}$$

&c. &c.

$$\text{Hence} \quad \frac{q_{n-1}}{q_n} = \cfrac{1}{a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \ldots \cfrac{1}{a_{n-m} + \cfrac{q_{n-m-2}}{q_{n-m-1}}}}}} .$$

Put $m = n-2$, and observe that $\frac{q_0}{q_1} = \frac{1}{a_1}$, and we find

$$\frac{q_{n-1}}{q_n} = (a_n \ldots \ldots a_1).$$

Corollary. Since $p_n = a_n p_{n-1} + p_{n-2}$, by following the same steps we find

$$\frac{p_{n-1}}{p_n}=\cfrac{1}{a_n+\cfrac{1}{a_{n-1}+\ldots+\cfrac{1}{a_{n-m}+\cfrac{p_{n-m-2}}{p_{n-m-1}}}}}.$$

Make $m=n-3$, and observe that $\frac{p_1}{p_2}=\frac{1}{a_2}$, hence

$$\frac{p_{n-1}}{p_n}=\cfrac{1}{a_n+\cfrac{1}{a_{n-1}+\ldots+\cfrac{1}{a_3+\cfrac{1}{a_2}}}}=(a_n\ldots a_2)$$

Corollary (2). Since $(a_n\ldots a_1)=\frac{q_{n-1}}{q_n}$

and $(a_n\ldots a_2)=\frac{p_{n-1}}{p_n}$.

Therefore $\frac{(a_n\ldots a_1)}{(a_n\ldots a_2)}=\frac{p_n}{q_n}\cdot\frac{q_{n-1}}{p_{n-1}}=\frac{(a_1\ldots a_n)}{(a_1\ldots a_{n-1})}$.

(92.) Conversion of algebraical formulæ into continued fractions.

1. To convert $\left(\frac{x+1}{x}\right)^2$ into a continued fraction, x being greater than 2:

First, when x is of the form $2y$, then $4y^2$ cannot be contained more than once in $4y^2+4y+1$, for 2 is greater than the greatest root of the equation $4y^2-4y-1=0$; therefore 2, or any quantity greater than 2, will render $4y^2>4y+1$, and therefore $8y^2>4y^2+4y+1$; consequently $4y^2$ cannot be contained twice or oftener in $4y^2+4y+1$: taking therefore 1 as the first quotient, the remainder is $4y+1$.

Again, $4y+1$ is evidently not contained y times in $4y^2$, but trying $y-1$ as the quotient we find a positive remainder, which is $3y+1$.

Next $3y+1$ is only contained once in $4y+1$, and the remainder is y, but y is contained 3 times in $3y+1$, with the remainder unity which terminates the operation, the whole of which is as follows:

$$\begin{array}{l}
4y^2)\,4y^2+4y+1\,(1\\
\quad\underline{4y^2\qquad\qquad}\\
\qquad\quad 4y+1)\,4y^2\quad(y-1\\
\qquad\qquad\quad\underline{4y^2-3y-1}\\
\qquad\qquad\qquad\quad 3y+1)\,4y+1\,(1\\
\qquad\qquad\qquad\qquad\qquad\underline{3y+1}\\
\qquad\qquad\qquad\qquad\qquad\; y)\,3y+1\,(3\\
\qquad\qquad\qquad\qquad\qquad\quad\underline{3y}\\
\qquad\qquad\qquad\qquad\qquad\qquad 1)\,\underline{y}\,(y;\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}$$

the continued fraction required is therefore

$$\left(\frac{2y+1}{2y}\right)^2=1+(y-1,\ 1,\ 3,\ y)$$

$$=1+\cfrac{1}{y-1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{y}}}}.$$

Secondly, when x is odd the proposed fraction is then of the form $\left(\frac{2y}{2y-1}\right)^2$, and the following operation will be easily understood.

$$\begin{array}{l}
4y^2-4y+1)4y^2\quad(1\\
\qquad\qquad\quad 4y^2-4y+1\\
\qquad\qquad\quad 4y-1)4y^2-4y+1(y-1\\
\qquad\qquad\qquad\qquad 4y^2-5y+1\\
\qquad\qquad\qquad\qquad\quad y)4y-1(3\\
\qquad\qquad\qquad\qquad\qquad 3y\\
\qquad\qquad\qquad\qquad\qquad y-1)y(1\\
\qquad\qquad\qquad\qquad\qquad\qquad y-1\\
\qquad\qquad\qquad\qquad\qquad\qquad 1)y-1(y-1\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

therefore $\left(\frac{2y}{2y-1}\right)^2=1+(y-1,\ 3,\ 1,\ y-1)$.

2. To convert $\left(\frac{6y+1}{6y}\right)^3$ into a continued fraction,

$$\begin{array}{l}
216y^3)216y^3+108y^2+18y+1(1\\
108y^2+18y+1)216y^3\qquad(2y-1\\
\qquad\qquad\qquad 216y^3-72y^2-16y-1\\
\qquad\qquad\qquad 72y^2+16y+1)108y^2+18y+1(1\\
\qquad\qquad\qquad\qquad\qquad 72y^2+16y+1\\
\qquad\qquad\qquad\qquad\qquad 36y^2+2y)72y^2+16y+1(2\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 72y^2+4y\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 12y+1)36y^2+2y(3y-1\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 36y^2-9y-1\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 11y+1)12y+1(1\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 11y+1\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad y)11y+1(11\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 11y\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1)y(y\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}$$

Hence $\left(\frac{6y+1}{6y}\right)^3=1+(2y-1,\ 1,\ 2,\ 3y-1,\ 1,\ 11,\ y)$.

3. To convert $\dfrac{\alpha^x - \beta^x}{\alpha^{x+1} - \beta^{x+1}}$ into a continued fraction, α, β being any quantities connected by the equation $\alpha\beta = -1$.

First observe that
$$(\alpha+\beta)(\alpha^x - \beta^x) = (\alpha^{x+1} - \beta^{x+1}) + (\alpha^x\beta - \beta^x\alpha)$$
$$= (\alpha^{x+1} - \beta^{x+1}) - (\alpha^{x-1} - \beta^{x-1}),$$

the ordinary operation will be then represented thus

$$\begin{array}{l} \alpha^x - \beta^x)\,\alpha^{x+1} - \beta^{x+1} \;.\;.\;.\;.\;.\;.\;.\; (\alpha+\beta \\ \qquad\quad \underline{\alpha^{x+1} - \beta^{x+1} - \alpha^{x-1} + \beta^{x-1}} \\ \qquad\quad \alpha^{x-1} - \beta^{x-1})\;\; \alpha^x \;\; - \beta^x \quad (\alpha+\beta, \end{array}$$

when an exactly similar operation recommences; therefore $\alpha+\beta$ is always the quotient, hence

$$(\alpha+\beta,\ \alpha+\beta,\ \alpha+\beta \ldots\ldots x \text{ times}) = \frac{\alpha^x - \beta^x}{\alpha^{x+1} - \beta^{x+1}},$$

or if $\alpha+\beta = a$, since $\alpha\beta = -1$, therefore $\alpha - \beta = 2\sqrt{\left(1+\frac{a^2}{4}\right)}$,

therefore,

$$(a, a, a \ldots x \text{ times}) = \frac{\left\{\frac{a}{2} + \sqrt{\left(\frac{a^2}{4}+1\right)}\right\}^x - \left\{\frac{a}{2} - \sqrt{\left(\frac{a^2}{4}+1\right)}\right\}^x}{\left\{\frac{a}{2} + \sqrt{\left(\frac{a^2}{4}+1\right)}\right\}^{x+1} - \left\{\frac{a}{2} - \sqrt{\left(\frac{a^2}{4}+1\right)}\right\}^{x+1}},$$

which converges to $\frac{1}{\alpha}$, or $-\beta$, that is to the least root of the equation $z^2 + az = 1$.

(93.) Problem. To find the values of finite and indefinite periodical continued fractions.

$$\text{Let } u_1 = \cfrac{1}{\alpha + \cfrac{1}{\beta}} \qquad u_2 = \cfrac{1}{\alpha + \cfrac{1}{\beta + \cfrac{1}{\alpha + \cfrac{1}{\beta,}}}}$$

and generally $u_x = (\alpha,\ \beta,\ \alpha,\ \beta,\ \alpha,\ \beta \ldots x \text{ times})$.

$$\text{Then } u_{x+1} = \cfrac{1}{\alpha + \cfrac{1}{\beta + u_x}} = \frac{\beta + u_x}{1 + \alpha\beta + \alpha u_x};$$

to satisfy which equation suppose

$$u_x = \frac{Am^x + Bn^x}{a \,.\, m^x + b \,.\, n^x},$$

$$\text{whence} \quad u_{x+1} = \frac{Am \,.\, m^x + Bn \,.\, n^x}{am \,.\, m^x + bn \,.\, n^x};$$

but if we substitute the assumed value of u_x in the former expression for u_{x+1}; clearing away the denominator $a \,.\, m^x + b \,.\, n^x$, we find also

$$u_{x+1} = \frac{(\mathrm{A}+\beta\, a) \,.\, m^x + (\mathrm{B}+\beta\, b) \,.\, n^x}{(a+a\alpha\beta+\mathrm{A}\beta)m^x+(b+b\alpha\beta+\mathrm{B}\beta) \,.\, n^x}$$

and by the comparison of similar terms we have the four following equations,

$$\begin{aligned} \mathrm{A}+\beta a &= m\mathrm{A} \\ \mathrm{B}+\beta b &= n\,\mathrm{B} \\ a+\alpha(\mathrm{A}+\beta a) &= m\,a \\ b+\alpha(\mathrm{B}+\beta b) &= n\,b\,; \end{aligned}$$

from the first and third of which

$$\frac{a}{\mathrm{A}} = \frac{m-1}{\beta} = \frac{m}{m-1} \,.\, \alpha,$$

and similarly from the second and fourth,

$$\frac{b}{\mathrm{B}} = \frac{n-1}{\beta} = \frac{n}{n-1} \,.\, \alpha,$$

which shew that m and n are two roots of the equation;

$$(m-1)^2 = m\,\alpha\,\beta$$

that is $\quad m+n = 2+\alpha\beta \;;\; mn = 1.$

Two more equations are necessary for the determination of the constants a, b; from which the values of A, B, may be deduced by the preceding equations.

Let therefore $x=1$, and we find $u_1 = \dfrac{\beta}{\alpha\beta+1} = \dfrac{\mathrm{A}\,m+\beta\,n}{a\,m+b\,n}$,

which give the two requisite equations, viz.

$$\begin{aligned} \mathrm{A}\,m+\mathrm{B}\,n &= \beta \\ a\,m+b\,n &= 1+\alpha\beta. \end{aligned}$$

Putting now for A its value $\dfrac{1}{m-1} \,.\, a\beta$, and $\dfrac{1}{n-1} \,.\, b\beta$ instead of B, the first equation becomes

$$a \,.\, \frac{m}{m-1} + b \,.\, \frac{n}{n-1} = 1,$$

from which we readily find

$$b = \frac{(n-1)\;(2+\alpha\beta-m)}{n\;(n-m)} = \frac{n-1}{n-m}$$

$$a = \ldots\ldots\ldots\ldots\ldots\ldots = \frac{m-1}{m-n};$$

therefore $\quad \mathrm{B} = \dfrac{\beta}{n-m} \qquad \mathrm{A} = \dfrac{\beta}{m-n} = -\mathrm{B}$

whence finally

$$u_x = \beta \cdot \frac{m^x - n^x}{(m-1)m^x - (n-1) \cdot n^x} = \beta \cdot \frac{m^x - m^{-x}}{(m^{x+1} - m^{-(x+1)}) - (m^x - m^{-x})}.$$

Example. Let $m=2$ $\beta=1$ $x=3$,

hence
$$\frac{126}{129} = \cfrac{1}{\frac{1}{2} + \cfrac{1}{1 + \cfrac{1}{\frac{1}{2} + \cfrac{1}{1 + \cfrac{1}{\frac{1}{2} + \frac{1}{1}}}}}}.$$

Again. Let $m + \frac{1}{m} = 2 \operatorname{Cos} \theta$ $\beta=1$

then

$$\frac{\operatorname{Sin}(x\theta)}{\operatorname{Sin}(x+1)\theta - \operatorname{Sin} x\theta} = \cfrac{1}{-4\operatorname{Sin}^2\frac{\theta}{2} + \cfrac{1}{1 + \cfrac{1}{-4\operatorname{Sin}^2\frac{\theta}{2} + \cfrac{1}{1 + (x \text{ periods})}}}}.$$

Lastly. Let $x = \infty$, then supposing $m > n$

$$u_\infty = \frac{\beta}{m-1},$$

or,

$$(\alpha, \beta, \alpha, \beta, \alpha, \beta \ldots . \textit{ad inf.}) = \frac{1}{\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha^2}{4} + \frac{\alpha}{\beta}\right)}} = \sqrt{\left(\frac{\beta^2}{4} + \frac{\beta}{\alpha}\right)} - \frac{\beta}{2}.$$

This result may be verified by making $u_\infty = z$, and then substituting z for all the periodical fractions after the first, we thus have

$$z = \cfrac{1}{\alpha + \cfrac{1}{\beta + z}} = \frac{\beta + z}{1 + \alpha\beta + \alpha z};$$

therefore $\alpha z^2 + \alpha\beta z = \beta$

$$z = -\frac{\beta}{2} \pm \sqrt{\left(\frac{\beta^2}{4} + \frac{\beta}{\alpha}\right)};$$

and since the continued fraction is positive, the upper sign of the surd must be used.

(94.) Let us next consider a continued fraction which recurs in x, complete periods, the number of partial denominators in each period being s; and the value of the continued fraction to the end of the xth period being denominated u_x, we have to seek the relation subsisting between u_{x+1} and u_x.

Represent by $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_s$, the partial denominators taken in the order in which they stand from the beginning to the end of a period, so that

$$u_1 = \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \ldots\ldots + \cfrac{1}{\alpha_s}}}} = (\alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_s)$$

then on account of the supposed recurrence of a similar period, we have

$$\begin{aligned} u_2 &= (\alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_{s-1}, \alpha_s + u_1) \\ u_3 &= (\alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_{s-1}, \alpha_s + u_2) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_{x+1} &= (\alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_{s-1}, \alpha_s + u_x). \end{aligned}$$

Suppose that the converging fractions to u_1 are successively

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3} \ldots\ldots\ldots\ldots \frac{p_s}{q_s};$$

or, $(\alpha_1) = \dfrac{p_1}{q_1}$ $(\alpha_1, \alpha_2) = \dfrac{p_2}{q_2}$ $(\alpha_1, \alpha_2, \alpha_3) = \dfrac{p_3}{q_3}$.

Hence $\quad u_1 = \dfrac{p_s}{q_s}$,

and by the known law of the formation of these fractions,

$$\begin{aligned} p_s &= \alpha_s p_{s-1} + p_{s-2} \\ q_s &= \alpha_s q_{s-1} + q_{s-2}. \end{aligned}$$

Now we may observe that the expression given above for u_{x+1}, differs only from the expression for u_1, in having $\alpha_s + u_x$ instead of α_s,

and since $\quad u_1 = \dfrac{\alpha_s p_{s-1} + p_{s-2}}{\alpha_s q_{s-1} + q_{s-2}}$,

therefore $\quad u_{x+1} = \dfrac{(\alpha_s + u_x)\, p_{s-1} + p_{s-2}}{(\alpha_s + u_x)\, q_{s-1} + q_{s-2}}$;

which formula may be further simplified by putting p_s instead of $\alpha_s p_{s-1} + p_{s-2}$ and q_s, instead of $\alpha_s q_{s-1} + q_{s-2}$, to which we have seen they are respectively equal, whence we have the equation sought, viz.

$$u_{x+1} = \frac{p_s + u_x \cdot p_{s-1}}{q_s + u_x \cdot q_{s-1}}.$$

From this formula we are prepared to find the express value of u_x, by a method similar to that which we employed in the simple case, when each period contained only two partial denominators.

Corollary 1. When the number of partial denominators, s, is exceedingly great, then the value of the continued fraction for one period only is the same as that for x periods.

For then since $\frac{p_{s-1}}{q_{s-1}} = \frac{p_s}{q_s}$;

therefore if $\qquad p_{s-1} = c\,p_s$, $\quad$ we have $q_{s-1} = c\,q_s$,

which substituted in the preceding equation, gives

$$u_{x+1} = \frac{p_s + c\,u_x p_s}{q_s + c\,u_x q_s} = \frac{p_s}{q_s} = u_1\,.$$

So that the hypothesis of a continued fraction commencing to recur after an infinite number of terms, and so recurring, if necessary, an infinite number of times, will not alter its value from that obtained without such an hypothesis.

Corollary 2. Suppose the number of periods infinite, we have

$$u_\infty(q_s + u_\infty q_{s-1}) = p_s + u_\infty p_{s-1};$$

$$\text{or,}\qquad u_\infty^2 + \frac{q_s - p_{s-1}}{q_{s-1}}\,.\,u_\infty = \frac{p_s}{q_{s-1}};$$

the positive root of which is the value of the fraction continued to infinity; the partial denominators also being supposed positive.

The product of the roots $-\frac{p_s}{q_{s-1}}$ being negative, shews that one of them is positive and the other negative; and their sum $-\frac{q_s - p_{s-1}}{q_{s-1}}$ being also negative, shews that the negative root is the greater; therefore the continued fraction, *ad inf.*, gives the least root of the above equation.

Example. To find the value of a continued fraction recurring *ad inf.*, the value of a single period being $\frac{25}{36}$.

(95.) To find the value of a recurring continued fraction which contains x complete periods.

By the preceding article we have

$$u_{x+1} = \frac{p_s + u_x p_{s-1}}{q_s + u_x q_{s-1}}.$$

Let $\qquad u_x = \frac{Am^x + Bn^x}{am^x + bn^x}$;

therefore $\qquad u_{x+1} = \frac{(ap_s + Ap_{s-1})m^x + (bp_s + Bp_{s-1})n^x}{(aq_s + Aq_{s-1})\,.\,m^x + (bq_s + Bq_{s-1})\,.\,n^x}.$

Hence $\qquad ap_s + Ap_{s-1} = Am \qquad aq_s + Aq_{s-1} = am$

$\qquad\qquad bp_s + Bp_{s-1} = Bn \qquad bq_s + Bq_{s-1} = bn$;

therefore $$\frac{a}{A}=\frac{m-p_{s-1}}{p_s}=\frac{q_{s-1}}{m-q_s}$$

$$\frac{b}{B}=\frac{n-p_{s-1}}{p_s}=\frac{q_{s-1}}{n-q_s};$$

from whence it follows that m, n are the two roots of the equation

$$m^2-(p_{s-1}+q_s)m+(p_{s-1}q_s-p_sq_{s-1})=0;$$

where the absolute term $=\pm 1$, or $(-1)^s$.

Hence $$m+n=p_{s-1}+q_s; \quad mn=(-1)^s.$$

Let $x=1$, in order to determine the constants a, b or A, B:

$$u_1=\frac{p_s}{q_s}=\frac{Am+Bn}{am+bn};$$

therefore $$Am+Bn=p_s \qquad am+bn=q_s.$$

Put for A its value $\frac{p_s}{m-p_{s-1}}.a$, and $\frac{p_s}{n-p_{s-1}}.b$ instead of B;

therefore $$\frac{a}{m-p_{s-1}}.m+\frac{b}{n-p_{s-1}}.n=1,$$

and $$am+b.n=q_s;$$

whence $$bn\left(1-\frac{m-p_{s-1}}{n-p_{s-1}}\right)=q_s+p_{s-1}-m=n;$$

therefore $$b=\frac{n-p_{s-1}}{n-m}; \text{ whence } B=\frac{p_s}{n-m}.$$

Similarly $$a=\frac{m-p_{s-1}}{m-n} \qquad A=\frac{p_s}{m-n};$$

therefore $$u_x=p_s.\frac{m^x-n^x}{(m-p_{s-1}).m^x-(n-p_{s-1}).n^x}$$

$$=p_s.\frac{m^x-n^x}{(m^{x+1}-n^{x+1})-p_{s-1}(m^x-n^x)}$$

$$=p_s.\frac{m^x-n^x}{q_s(m^x-n^x)-(-1)^s(m^{x-1}-n^{x-1})}.$$

Whence this theorem.

If $\frac{p_s}{q_s}$ be the value of a single complete period, then that of x complete periods will be $\dfrac{p_s}{q_s\pm\dfrac{m^{x-1}-n^{x-1}}{m^x-n^x}}$, taking the upper or lower sign according as s is odd or even.

(96.) We propose, in the next place, to form the converging frac-

tions successively which express the values of complete periods of a recurring continued fraction.

Using the same notation as before, we have

$$u_1=\frac{p_s}{q_s} \quad u_2=\frac{p_{2s}}{q_{2s}} \quad u_3=\frac{p_{3s}}{q_{3s}} \ldots\ldots \quad u_x=\frac{p_{xs}}{q_{xs}}.$$

But by the last article we have also

$$u_x=\frac{p_s(m^x-n^x)}{q_s(m^x-n^x)-(-1)^s(m^{x-1}-n^{x-1})};$$

the numerator and denominator of which may be reduced to rational and integer formulæ by means of a common multiplier which we may represent by $\frac{1}{\lambda}$; hence, we find,

$$\lambda p_{xs}=p_s(m^x-n^x)$$

$$\lambda q_{xs}=q_s(m^x-n^x)-(-1)^s(m^{x-1}-n^{x-1}).$$

But since $\quad m^2-(p_{s-1}+q_s)m+(-1)^s=0,$

therefore $\quad m^{x+2}-(p_{s-1}+q_s)m^{x+1}+(-1)^s.m^x=0;$

and a similar equation is true for n. Therefore

$$p_{(x+2)s}=(p_{s-1}+q_s)p_{(x+1)s}+(-1)^{s-1}.p_{xs}$$

$$q_{(x+2)s}=(p_{s-1}+q_s)q_{(x+1)s}+(-1)^{s-1}.q_{xs}.$$

Thus we find successively

$$\begin{cases} p_{2s}=(p_{s-1}+q_s)p_s \\ q_{2s}=(p_{s-1}+q_s).q_s+(-1)^{s-1} \end{cases}$$

$$\begin{cases} p_{3s}=(p_{s-1}+q_s).p_{2s}+(-1)^{s-1}.p_s \\ q_{3s}=(p_{s-1}+q_s)q_{2s}+(-1)^{s-1}.q_s \end{cases}$$

&c. &c.

Example. To find the converging fractions to the successive periods of a continued recurring fraction of which one complete period is

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3}}}.$$

In this case $\quad s=3 \quad \frac{p_1}{q_1}=\frac{1}{1} \quad \frac{p_2}{q_2}=\frac{2}{3} \quad \frac{p_3}{q_3}=\frac{7}{10};$

the general formula above obtained is then (since $p_2+q_3=12$)

$$p_{(x+2)s}=12p_{(x+1)s}+p_{xs}$$

$$q_{(x+2)s}=12q_{(x+1)s}+q_{xs}.$$

Hence $\quad \frac{p_3}{q_3}=\frac{7}{10} \quad \frac{p_6}{q_6}=\frac{84}{121} \quad \frac{p_9}{q_9}=\frac{1015}{1462}$, &c.

Corollary. Between the expressions for $p_{(x+2)s}$, $q_{(x+2)s}$, eliminate the multiplier $p_{s-1}+q_s$, and we find

$$p_{(x+2)s}q_{(x+1)s}-p_{(x+1)s}q_{(x+2)s}=(-1)^s\{p_{(x+1)s}q_{xs}-p_{xs}q_{(x+1)s}\}.$$

Hence $p_x q_{(x-1)t} - p_{(x-1)t} q_{xt} = (-1)^{t(x-1)} . p_t$,

since the first of the converging fractions is $\frac{0}{1}$.

(97.) Theorem. Every recurring fraction of a complete number of periods may be converted into another of which each period consists of one figure only.

Retaining the same notation, we have, by the article preceding the last,

$$u_x = \cfrac{p_t}{q_t \pm \cfrac{m^{x-1} - n^{x-1}}{m^x - n^x}}.$$

But by Art. (92) $\frac{m^{x-1} - n^{x-1}}{m^x - n^x} = \cfrac{1}{m+n+\cfrac{1}{m+n+\cfrac{1}{m+n\ldots(x-1 \text{ times.})}}}$;

therefore $$u_x = p_t . \cfrac{1}{q_t \pm \cfrac{1}{p_{t-1}+q_t+\cfrac{1}{p_{t-1}+q_t+\cfrac{1}{p_{t-1}+q_t\ldots(x-1 \text{ times})}}}}.$$

Example. $$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3\ldots(x \text{ periods.})}}}}}}$$

$$= 7 . \cfrac{1}{10+\cfrac{1}{12+\cfrac{1}{12+\&c\ldots(x-1 \text{ times.})}}}$$

(98.) To find the value of a continued fraction commencing arbitrarily, but afterwards proceeding in recurring periods.

Let the arbitrary part which does not recur be $(\beta_1, \beta_2 \ldots \beta_t)$, and let each recurring period be $(\alpha_1, \alpha_2 \ldots \alpha_s)$; let $\frac{P_{t-2}}{Q_{t-2}}, \frac{P_{t-1}}{Q_{t-1}}, \frac{P_t}{Q_t}$ be the three last converging fractions to the first or non-recurring fraction; let u_x be the value of the recurring fraction taken for x complete periods; and let v_x be the corresponding or required value of the compound continued fraction proposed; then

$$v_x = (\beta_1, \beta_2, \beta_3 \ldots \beta_{t-1}, \beta_t + u_x).$$

Now
$$\frac{P_t}{Q_t}=(\beta_1, \beta_2, \beta_3 \ldots \beta_{t-1}, \beta_t)$$

$$=\frac{\beta_t P_{t-1}+P_{t-2}}{\beta_t Q_{t-1}+Q_{t-2}};$$

therefore
$$v_x=\frac{(\beta_t+u_x)P_{t-1}+P_{t-2}}{(\beta_t+u_x)Q_{t-1}+Q_{t-2}},$$

$$=\frac{P_t+u_x P_{t-1}}{Q_t+u_x Q_{t-1}}.$$

But we know that $u_x=\dfrac{p_s}{q_s \pm \dfrac{m^{x-1}-n^{x-1}}{m^x-n^x}}$;

Hence
$$v_x=\frac{(P_t q_s+P_{t-1}p_s)(m^x-n^x)\pm P_t(m^{x-1}-n^{x-1})}{(Q_t q_s+Q_{t-1}p_s)(m^x-n^x)\pm Q_t(m^{x-1}-n^{x-1})}.$$

Corollary. We may prove, as before, that the law of the formation of the fractions converging to whole periods is still the same.

$$\begin{cases} P_{t+s}=P_t q_s+P_{t-1}p_s \\ Q_{t+s}=Q_t q_s+Q_{t-1}.p_s \end{cases}$$

$$\begin{cases} P_{t+2s}=(q_s+p_{s-1}).P_{t+s}+(-1)^{s-1}.P_t \\ Q_{t+2s}=(q_s+p_{s-1}).Q_{t+s}+(-1)^{s-1}.Q_t, \end{cases}$$

&c. &c.

Example. To form the fractions converging to the periods of the compound continued fraction (4, 3, 2, 1, 2, 3, 1, 2, 3, &c.).

First, $t=3$ $\quad \dfrac{P_1}{Q_1}=\dfrac{1}{4}, \quad \dfrac{P_2}{Q_2}=\dfrac{3}{13}. \quad \dfrac{7}{30}=\dfrac{P_3}{Q_3},$

$s=3$ $\quad \dfrac{p_1}{q}=\dfrac{1}{1} \quad \dfrac{p_2}{q_2}=\dfrac{2}{3} \quad \dfrac{p_3}{q_3}=\dfrac{7}{10}.$

Hence,
$$P_{t+s}=P_t\, q_s+P_{t-1}\, p_s= \ 7\times 10+\ 3\times 7=\ 91,$$
$$Q_{t+s}=Q_t\, q_s+Q_{t-1}\, p_s=\ 30\times 10+13\times 7=391.$$

The remaining numerators are formed by multiplying that immediately preceding by $12=q_s+p_{s-1}$, and adding the numerator again preceding; and the denominators are found by the same law: the required converging fractions are therefore

$$\frac{7}{30},\ \frac{91}{391},\ \frac{1099}{4722},\ \frac{13279}{57055},\ \&c.$$

Corollary. It may be easily proved in a manner similar to that cited in the analogous cases which have already occurred, that

$$P_{t+xs}\, Q_{t+(x-1)s}-P_{t+(x-1)s}Q_{t+xs}=(-1)^{s(x-1)+t}.p_s.$$

Moreover, it is visible that the converging fractions thus obtained are

alternately greater and less than the true value; and the error from the value of the fraction continued to infinity is less than $\frac{p_s}{Q^2_{t+2s}}$.

(99.) *Scholium.* This method adds at each operation a number of accurate decimal places which cannot exceed double the number of digits in $(p_s+q_{s-1})^2$. I have sought a more rapid method of forming the converging fractions, which quadruples at each operation the number of accurate decimal places.

First, consider the fraction to complete periods, to which the converging fractions are $\frac{p_s}{q_s}\ \frac{p_{2s}}{q_{2s}}\ \frac{p_{3s}}{q_{3s}}$ &c., and when this fraction is inverted, let $\frac{p'_s}{q'_s}\ \frac{p'_{2s}}{q'_{2s}}\ \frac{p'_{3s}}{q'_{3s}}$ &c. be the converging fractions to the new continued fraction, which has also complete periods.

Then, as we have already seen,

$$p'_s=q_{s-1}\quad q'_s=q_s\quad p'_{2s}=q_{2s-1}\quad q'_{2s}=q_{2s}\ \text{\&c.}$$

But since $p_{2s}=(p_{s-1}+q_s)p_s$, and $p_{s-1}=\frac{p_s\,q_{s-1}\pm 1}{q_s}=\frac{p_sp'_s\pm 1}{q'_s}$; ∴

therefore $$p_{2s}=\frac{q_sq'_s+p_sp'_s\pm 1}{q_s}.p_s;$$

similarly, $$p'_{2s}=\frac{q_sq'_s+p_sp'_s\pm 1}{q'_s}.p'_s;$$

therefore $$\frac{p'_{2s}}{p_{2s}}=\frac{p'_s}{p_s}=\frac{q_{s-1}}{p_s}=m \text{ a constant.}$$

Hence, $$m=\frac{p'_s}{p_s}=\frac{p'_{2s}}{p_{2s}}=\frac{p'_{4s}}{p_{4s}}=\dots\ \frac{p'_{2^xs}}{p_{2^xs}}.$$

Hence, $$\left.\begin{aligned} p_{2s}&=\left(q_s+\frac{m.p^2_s\pm 1}{q_s}\right).p_s\\ q_{2s}&=\left(q_s+\frac{m.p^2_s\pm 1}{q_s}\right).q_s+(-1)^{s-1}\end{aligned}\right\}.$$

Write now $2s$ instead of s:

therefore, $$p_{4s}=\left(q_{2s}+m.\frac{p^2_{2s}+1}{q_{2s}}\right).p_{2s}$$

$$q_{4s}=\left(q_{2s}+m.\frac{p_{2s}^2+1}{q_{2s}}\right).q_{2s}-1;$$

and generally, $$p_{2^{x+1}s}=\left(q_{2^xs}+\frac{m.p^2_{2^xs}+1}{q_{2^xs}}\right).p_{2^xs},$$

$$q_{2^{x+1}s}=\left(q_{2^xs}+\frac{m.p^2_{2^xs}+1}{q_{2^xs}}\right).q_{2^xs}-1,$$

the same multipliers within the brackets give the formula for $P_{t+2^{x+1}s}$.

and $Q_{t+2^x+1,}$; and since it is evident that $q_{2^x+1,} > (q_{2^x,})^2$, therefore $(q_{2^x+1})^2 > (q_{2^x,})^4$, which shows that the number of accurate figures is at least quadrupled at each operation.

It is almost unnecessary to add, that the multipliers within the brackets are always integers, being the same as $q_{2^x,} + p_{2^x,-1}$;

$$P_{t+2^x,} = P_t . q_{2^x,} + P_{t-1} . p_{2^x,},$$
$$Q_{t+2^x,} = Q_t . q_{2^x,} + Q_{t-1} . p_{2^x,}.$$

Example. Reconsider the continued fraction

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+}}}}}} \text{ \&c.}$$

$$\frac{p_3}{q_3} = \frac{7}{10} \qquad q_3 = 3.$$

First multiplier $$= 10 + \frac{\frac{3}{7} . 7^2 - 1}{10} = 12;$$

therefore $$\frac{p_6}{q_6} = \frac{84}{121}.$$

Second multiplier $$= 121 + \frac{\frac{3}{7} . 84^2 + 1}{121} = 146$$

$$\frac{p_{12}}{q_{12}} = \frac{84 \times 146}{121 \times 146 - 1} = \frac{12264}{17665}.$$

Third multiplier $$= 17665 + \frac{\frac{3}{7}(12264)^2 + 1}{17665} = 21214$$

$$\frac{p_{24}}{q_{24}} = \frac{12264 \times 21214}{17665 \times 21214 - 1} = \frac{260168496}{374745209},$$

the error of which from the value of the fraction continued *ad inf.* must be less than $\dfrac{1}{(374745209)^2}$.

(100.) *Problem.* To find the value of a periodic continued fraction, not commencing with periodic terms, and continued to infinity.

Employing still the same notation, let U be the value of the periodic part continued to infinity, and V the sought value of the whole.

Let m be the greatest root of the equation

$$m^2-(p_{t-1}+q_t)m+(-1)^t=0;$$

then by the foregoing articles we have

$$U=\frac{p_t}{q_t-\dfrac{(-1)^t}{m}}=\frac{p_t}{q_t-n}$$

whence $$n=q_t-\frac{p_t}{U}.$$

Moreover, since $$V=\frac{P_t+UP_{t-1}}{Q_t+UQ_{t-1}};$$

therefore $$U=-\frac{VQ_t-P_t}{VQ_{t-1}-P_{t-1}};$$

therefore $$n=q_t+p_t.\frac{VQ_{t-1}-P_{t-1}}{VQ_t-P_t}.$$

Now since $$n^2-(p_{t-1}+q_t)n+(-1)^t=0,$$

therefore $$(n-q_t)^2+(q_t-p_{t-1})(n-q_t)-q_{t-1}p_t=0;$$

in which, if we substitute the value of $n-q_t$ in terms of V, and multiply the resulting equation by $\dfrac{(VQ_t-P_t)^2}{p_t}$, we find

$$p_t(VQ_{t-1}-P_{t-1})^2+(q_t-p_{t-1})(VQ_{t-1}-P_{t-1})(VQ_t-P_t)-q_{t-1}(VQ_t-P_t)^2=0;$$

therefore,

$$\{p_tQ^2_{t-1}+(q_t-p_{t-1})Q_tQ_{t-1}-q_{t-1}Q^2_t\}.V^2$$
$$-\{2p_tP_{t-1}Q_{t-1}+(q_t-p_{t-1})(P_tQ_{t-1}+Q_tP_{t-1})-2q_{t-1}.P_tQ_t\}.V$$
$$+\{p_tP^2_{t-1}+(q_t-p_{t-1}).P_tP_{t-1}-q_{t-1}.P^2_t\}=0.$$

Lagrange has obtained the same equation by a very different process. (*Vide Traité de la Résolution des Equations*, p. 56.)

By referring to the quadratic, which is arranged according to the powers of $n-q_t$, we see that the absolute term is negative; therefore $n-q_t$ and $m-q_t$, which are its roots, must have contrary signs; therefore, $n-q_t$ must be negative, m being greater than n, consequently $\dfrac{VQ_{t-1}-P_{t-1}}{VQ_t-P_t}$ must be negative for that value of V which corresponds to the continued fraction, the other root which is foreign would make the same fraction positive; one root of the equation in V must therefore lie between $\dfrac{P_{t-1}}{Q_{t-1}}$ and $\dfrac{P_t}{Q_t}$, and is that which is sought in this question; the other root does not lie between the same limits: we thus know which root it is proper to reject, and which to retain.

Example. Find the value of the indefinite continued fraction:

$$\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3}+\&c.}}}}}}}$$

The non-recurring part consists of two places, $t=2$;

therefore $\qquad \frac{P_{t-1}}{Q_{t-1}}=\frac{1}{4} \qquad \frac{P_t}{Q_t}=\frac{3}{13}.$

The recurring periods contain each three places, $s=3$;

hence $\qquad \frac{p_{s-1}}{q_{s-1}}=\frac{2}{3} \qquad \frac{p_s}{q_s}=\frac{7}{10}.$

Therefore, $239V^2-103V+11=0$;

$$V=\frac{103\pm\sqrt{93}}{478},$$

the sign + must be given to the radical to obtain that root which lies between $\frac{1}{4}$ and $\frac{3}{13}$, which is then the true value of the continued fraction.

(101.) To extract the square root of a number in the form of a continued fraction.

Rule. Take the nearest integer immediately below the true square root of the number, and put the sought square root equal to this number + the sought square root *minus* this number, and transpose the surd to the denominator by multiplying this difference by a similar sum, as $\sqrt{5}-1$ would be changed to $\frac{4}{\sqrt{5}+1}$.

Take the nearest integer to the denominator thus formed divided by the numerator, and then by subtraction make the proper compensation, and let the surd thus arising be again transferred to the denominator in similar manner; for example $\sqrt{3}=1+\frac{\sqrt{3}-1}{1}=1+\frac{2}{\sqrt{3}+1}$.

By continuing this process all the partial denominators of the continued fraction required will be found.

Thus, $\qquad \sqrt{17}=4+\frac{\sqrt{17}-4}{1}=4+\frac{1}{\sqrt{17}+4},$

$$\sqrt{17}+4=8+\frac{\sqrt{17}-4}{1}=8+\frac{1}{\sqrt{17}+4},$$

$$\sqrt{17}+4=\ldots\ldots\ldots=8+\frac{1}{\sqrt{17}+4},$$

&c. &c.

therefore, $\sqrt{17}=4+\cfrac{1}{8+\cfrac{1}{8+\cfrac{1}{8}\ \&c.}}$

To find $\sqrt{21}$.

$$\sqrt{21}=4+\frac{\sqrt{21}-4}{1}=4+\frac{5}{\sqrt{21}+4},$$

$$\frac{\sqrt{21}+4}{5}=1+\frac{\sqrt{21}-1}{5}=1+\frac{4}{\sqrt{21}+1}$$

$$\frac{\sqrt{21}+1}{4}=1+\frac{\sqrt{21}-3}{4}=1+\frac{3}{\sqrt{21}+3}$$

$$\frac{\sqrt{21}+3}{3}=2+\frac{\sqrt{21}-3}{3}=2+\frac{4}{\sqrt{21}+3}$$

$$\frac{\sqrt{21}+3}{4}=1+\frac{\sqrt{21}-1}{4}=1+\frac{5}{\sqrt{21}+1}$$

$$\frac{\sqrt{21}+1}{5}=1+\frac{\sqrt{21}-4}{5}=1+\frac{1}{\sqrt{21}+4}$$

$$\sqrt{21}+4=8+\frac{\sqrt{21}-4}{1}=8+\frac{5}{\sqrt{21}+4}$$

$$\frac{\sqrt{21}+4}{5}=1+\frac{\sqrt{21}-1}{5}=1+\frac{4}{\sqrt{21}+1}$$

$$\frac{\sqrt{21}+1}{4}=1+\frac{\sqrt{21}-3}{4}=1+\frac{3}{\sqrt{21}+3}$$

After which the same formula recur, therefore

$$\sqrt{21}=4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\ \&c.}}}}}}}}}}$$

and the converging fractions may be found by the method pointed out in Art. (99.)

To extract the square root of a fraction in the form of a continued fraction.

Example. To find the square root of $\frac{5}{3}$.

$$\sqrt{\frac{5}{3}}=1+\frac{\sqrt{\frac{5}{3}}-1}{1}=1+\frac{\frac{2}{3}}{\sqrt{\frac{5}{3}}+1}=1+\frac{2}{\sqrt{15}+3}$$

$$\frac{\sqrt{15}+3}{2}=3+\frac{\sqrt{15}-3}{2}=3+\frac{3}{\sqrt{15}+3}$$

$$\frac{\sqrt{15}+3}{3}=2+\frac{\sqrt{15}-3}{3}=2+\frac{2}{\sqrt{15}+3}$$

after which the periods recur; therefore

$$\sqrt{\left(\frac{5}{3}\right)}=1+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{3+\&c.}}}}}}$$

Again, to find the square root of $\frac{1}{2}$.

$$\sqrt{2}=1+\frac{\sqrt{2}-1}{1}=1+\frac{1}{\sqrt{2}+1}$$

$$\sqrt{2}+1=2+\frac{\sqrt{2}-1}{1}=2+\frac{1}{\sqrt{2}+1}$$

after which the periods are the same, therefore

$$\sqrt{\left(\frac{1}{2}\right)}=\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}\ \&c.}}}$$

Again, to extract the square root of $\frac{4}{7}$.

$$\sqrt{\left(\frac{7}{4}\right)}=1+\frac{\sqrt{\frac{7}{4}}-1}{1}=1+\frac{\frac{3}{4}}{\sqrt{\frac{7}{4}}+1}=1+\frac{3}{\sqrt{28}+4}$$

$$\frac{\sqrt{28}+4}{3}=3+\frac{\sqrt{28}-5}{3}=3+\frac{1}{\sqrt{28}+5}$$

$$\frac{\sqrt{28}+5}{1}=10+\frac{\sqrt{28}-5}{1}=10+\frac{3}{\sqrt{28}+5}$$

$$\frac{\sqrt{28}+5}{3}=3+\frac{\sqrt{28}-4}{3}=3+\frac{4}{\sqrt{28}+4}$$

$$\frac{\sqrt{28}+4}{4}=2+\frac{\sqrt{28}-4}{4}=2+\frac{3}{\sqrt{28}+4}$$

after which the periods recur,

therefore, $$\sqrt{\frac{4}{7}}=\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{10+\cfrac{1}{3+\cfrac{1}{2+}}}}}\ \&c.$$

A similar method is applicable to the solution of quadratic equations to express the positive roots under the form of continued fractions.

Example. $$2x^2-3x=6.$$

$$x=\frac{3}{4}+\frac{\sqrt{57}}{4}$$

Now, $$\frac{3+\sqrt{57}}{4}=2+\frac{\sqrt{57}-5}{4}=2+\frac{8}{\sqrt{57}+5}$$

$$\frac{5+\sqrt{57}}{8}=1+\frac{\sqrt{57}-3}{8}=1+\frac{6}{\sqrt{57}+3}$$

$$\frac{3+\sqrt{57}}{6}=1+\frac{\sqrt{57}-3}{6}=1+\frac{8}{\sqrt{57}+3}$$

$$\frac{3+\sqrt{57}}{8}=1+\frac{\sqrt{57}-5}{8}=1+\frac{4}{\sqrt{57}+5}$$

$$\frac{5+\sqrt{57}}{4}=3+\frac{\sqrt{57}-7}{4}=3+\frac{2}{\sqrt{57}+7}$$

$$\frac{7+\sqrt{57}}{2}=7+\frac{\sqrt{57}-7}{2}=7+\frac{4}{\sqrt{57}+7}$$

$$\frac{7+\sqrt{57}}{4}=3+\frac{\sqrt{57}-5}{4}=3+\frac{8}{\sqrt{57}+5}$$

after which the partial denominators recur, therefore,

$$x=2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\&c.}}}}}}}}$$

The successive converging fractions to the first period are

$$\frac{1}{1},\ \frac{1}{2},\ \frac{2}{3},\ \frac{7}{11},\ \frac{51}{80},\ \frac{160}{251}$$

Hence, $\dfrac{p_6}{q_6}=\dfrac{160}{251}$ $q_5=80$, therefore, $\dfrac{q_5}{p_6}=\dfrac{1}{2}$.

First multiplier $=251+\dfrac{\frac{1}{2}.160^2+1}{251}=302$;

therefore, $\dfrac{p_{12}}{q_{12}}=\dfrac{160\times302}{251\times302-1}=\dfrac{48320}{75801}$.

Second multiplier, $=75801+\dfrac{\frac{1}{2}.48320^2+1}{75801}=91202$.

Hence, $\dfrac{p_{24}}{q_{24}}=\dfrac{48320\times91202}{75801\times91202-1}=\dfrac{4406880640}{6913202801}$;

and this fraction increased by 2, errs from the true value of x by a quantity which is less than $\dfrac{1}{40,000,000,000,000,000,000}$.

(102) *Theorem.* In applying the method of continued fractions to the solution of equations, the transformed equations after a few at the beginning will have uniformly their first and last terms affected with contrary signs.

Let $\phi(x)=0$ be the proposed equation, and λ be the nearest integer below one of its roots, then put $x=\lambda+\dfrac{1}{y}$; the transformed equation in y has as many positive roots greater than unity, as there are values of x included between the consecutive numbers λ and $\lambda+1$.

Again, it is possible that two or more values of y may lie between two successive integers, λ' and $\lambda'+1$; in which case the transformed equation arising by making $y=\lambda'+\dfrac{1}{z}$, has a corresponding number of positive roots greater than unity.

When, in continuing this process, we arrive at a transformed equation which has not two or more roots included between two consecutive numbers, as

$$au^m+bu^{m-1}+cu^{m-1}+\ldots\ldots+k=0;$$

then, making $u=s+\frac{1}{w}$, s being the nearest integer below a root of this equation, the transformed equation in w can have but one positive root greater than unity, and then all the consecutive transformed equations will be in the same condition.

Represent the equation in w by $F(w)=0$; let l be the nearest integer below the only positive root it has greater than unity, and let $w=l+\frac{1}{t}$.

Then,

$$F\left(l+\frac{1}{t}\right)=\frac{1}{t^m}\left\{F(l).t^m+F'(l).t^{m-1}+\frac{F''(l)}{1.2}.t^{m-2}+..\frac{F^{\prime\prime(m)}(l)}{1.2..m}\right\}=0,$$

therefore the first term in the transformed equation would be $F(l).t^m$, and the last $\frac{F^{\prime\prime\prime(m)}(l)}{1.2.3....m}$; these terms must have contrary signs.

For, by supposition, there is but a single root of the equation, $F(w)=0$ between the limits l and $+\infty$; therefore, $F(l)$ and $F(\infty)$ have contrary signs, or $F\left(l+\frac{1}{t}\right)$ must have contrary signs for $t=0$ and $t=\infty$; now, $t^mF\left(l+\frac{1}{t}\right)=\frac{F^{\prime\prime\prime(m)}(l)}{1.2.3...m}$ when $t=0$; this, therefore, must have a contrary sign to that which affects $F(l)$.

Again, since the equation in t has but one positive root between $+1$ and ∞, therefore the same reasoning applies to show that the next transformed equation and all the succeeding ones, have their first and last terms affected with contrary signs.

Theorem. The root of every quadratic equation expressed in the form of a continued fraction is necessarily recurring.

Suppose the transformations to be continued until the first and last terms have contrary signs, which must eventually happen, as we have seen in the last article; let then the equation be

$$a'x^2-2bx-a=0.$$

Let λ be the nearest integer below the sole positive root which this equation has, and make $x=\lambda+\frac{1}{x'}$;

therefore, $\quad a''x'^2-2b'x'-a'=0,$

where $\quad \left.\begin{array}{l} a''=-a'\lambda^2+2b\lambda+a \\ b'=+a'\lambda-b \end{array}\right\}$ changing all the signs that the absolute term may be negative;

from whence we find

$$b'^2+a'a''=b^2+aa'$$

similarly, if the next transformed equation be represented by

$$a'''x''^2-2b''x''-a''=0,$$

we must have $b''^2+a''a'''=b'^2+a'a''=b^2+aa'$;

in fact this is the quantity which is under the radical sign in the actual

solution of the primitive equation; and it is manifest that this surd ought to remain unchanged in all the transformed equations, since the roots of these different equations are rational functions of each other.

Put, therefore, $b^2+aa'=C$,

then, since $b'^2+a'a''=C$, and a', a'', b', are integers,

therefore, $b'<\sqrt{C}$, and $a'<C$.

Therefore the coefficients a', b', a'', b'', &c., cannot in any of the transformed equations exceed given integers; and consequently, after a sufficient number of transformations, the same co-existing system of values for a' and b' must recur; and therefore, also, the same value for $a''=\dfrac{C-b'^2}{a'}$; the transformed equation thus arrived at being the same as one obtained before in the process, it follows that the whole operation, and therefore the periods of the continued fraction, must recur.

(103.) *Problem.* To convert a given continued fraction into a series.

Let $\dfrac{p_1}{q_1}$, $\dfrac{p_2}{q_2}$, $\dfrac{p_3}{q_3}$, &c., be the converging fractions to a given continued fraction, and $\dfrac{P}{Q}$ the value when continued to infinity.

Then,

$$\frac{p_2}{q_2}-\frac{p_1}{q_1}=\frac{p_2q_1-p_1q_2}{q_1q_2}=\frac{-1}{q_1q_2}$$

$$\frac{p_3}{q_3}-\frac{p_2}{q_2}=\dots\dots\dots\dots\frac{1}{q_2q_3}$$

$$\frac{p_4}{q_4}-\frac{p_3}{q_3}=\dots\dots\dots\dots\frac{-1}{q_3q_4}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}=\dots\dots\dots\dots\frac{(-1)^{n-1}}{q_{n-1}q_n}$$

Hence, by addition,

$$\frac{p_n}{q_n}=\frac{p_1}{q_1}-\frac{1}{q_1q_2}+\frac{1}{q_2q_3}-\frac{1}{q_3q_4}+\dots\dots+\frac{(-1)^{n-1}}{q_{n-1}q_n},$$

and $$\frac{P}{Q}=\frac{p_1}{q_1}-\frac{1}{q_1q_2}+\frac{1}{q_2q_3}-\text{ &c., } ad\ inf.$$

The terms of this series continually diminish, and are alternately positive and negative; therefore the error committed by taking n terms of the series, instead of the whole, is less than the $(n+1)$th term. But in some cases it is convenient to take some of the partial denominators negative.

Thus, the continued fraction which represents the ratio of the circumference to the diameter of a circle may be written either as

$$3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\text{ \&c.}}}}}}}}}$$

or

$$3+\cfrac{1}{7+\cfrac{1}{16-\cfrac{1}{294-\cfrac{1}{3-\cfrac{1}{3+\text{ \&c.}}}}}}$$

the second of which is best adapted to give a converging series, which, formed after the formula above given, is as follows:

$$\pi=3+\frac{1}{7}-\frac{1}{7\times 113}-\frac{1}{113\times 33215}-\frac{1}{33215\times 99532}-\text{ \&c.}$$

Periodic continued fractions may be converted into still more convergent series by a similar process.

(104.) To determine the commencement of the periods when a quadratic equation is solved in the form of a continued fraction.

We have seen that in the process of the successive transformations of the proposed equation we arrive at one of the form

$$a'x^2-2bx-a=0,$$

where a and a' have necessarily the same sign, and that thenceforward all the successive quantities a', a'', a''', &c., have the same sign. One of the two commencing equations of this form, namely, that in which the last term (a or a' as the case may be) is less than $\sqrt{C}$, (since $aa'=C-b^2$) will afterwards re-appear, and therefore the fraction will be periodical from this point of commencement.

For suppose a to be that which is less than $\sqrt{C}$, then the series of numbers a, a', a'', &c., b, b', b'', &c. will all have the same sign as $\sqrt{C}$.

For the equation above written gives

$$x=\frac{b+\sqrt{C}}{a'};$$

and by the nature of the transformations it is obvious that $x>1$.

Therefore, $$\frac{b+\sqrt{C}}{a'}>1;$$

hence, $$\frac{C-b^2}{a'} \cdot \frac{1}{\sqrt{C}-b} > 1.$$

But since $$C = aa' + b^2,$$

therefore, $$\frac{a}{\sqrt{C}-b} > 1; \text{ similarly } \frac{a'}{\sqrt{C}-b'} > 1, \&c.$$

But since $b < \sqrt{a\,a' + b^2}$, therefore $\sqrt{C}-b$ has the same sign as $\sqrt{C}$; wherefore a must necessarily have the same sign; and like reasoning applies to the quantities, a', a'', &c.

Now, by hypothesis, $a < \sqrt{C}$, and it is proved that $\frac{a}{\sqrt{C}-b} > 1$; therefore b must have also the same sign as $\sqrt{C}$, for if it had a contrary sign the denominator of this fraction would be numerically greater than the numerator, and therefore the fraction would be less than unity.

Again, b' must have the same sign also, for, if it had a contrary sign, then, since $\lambda a' = b + b'$, λ being a whole number, and b, b', each $< \sqrt{C}$, it would follow that $a' = \frac{b-(-b')}{\lambda}$ is also $< \sqrt{C}$; whence, as before, b' must have the same sign as $\sqrt{C}$; therefore the supposition that it has a contrary sign is wrong; and the same reasoning may be extended to b'', b''', &c.

Now if any cotemporary coefficients in both these series as a''', b''', &c. are given, the preceding ones may be thence determined.

For since $\quad b''' = a'''\lambda'' - b'', \quad$ and $b'' < \sqrt{C}$,

therefore $$\lambda'' < \frac{b''' + \sqrt{C}}{a'''};$$

similarly, $$\lambda' < \frac{b'' + \sqrt{C}}{a''}.$$

But by the nature of the operation λ' is a positive integer;

therefore, $$b'' + \sqrt{C} > a''.$$

Now, $$a''a''' = C - b''^2 = (\sqrt{C} - b'')(\sqrt{C} + b''),$$
whence $$\sqrt{C} - b'' < a'''.$$

In this inequality put for b'' its value $a'''\lambda'' - b'''$; hence
$$\sqrt{C} + b''' - a'''\lambda'' < a''';$$

therefore $\lambda'' > \frac{b''' + \sqrt{C}}{a'''} - 1$; and it has been proved to be less than $\frac{b''' + \sqrt{C}}{a'''}$, and is a positive integer, therefore it can only be the nearest positive integer below $\frac{b''' + \sqrt{C}}{a'''}$, and is therefore a known

quantity; and from thence the quantities $b'' = a''' \lambda'' - b''$, and $a'' = \frac{C - b''^2}{a'''}$ are determined.

Now we have seen that in the successive transformed equations $a'x^2 - 2bx - a = 0$, $a''x'^2 - 2b'x - a' = 0$, $a'''x''^2 - 2b''x'' - a''$, &c., the coefficients of some pair, as (a'', b'''), $(a'''^{(n)}, b''^{(n)})$ must be the same, therefore the coefficients of the preceding pair $(a'' b'')$, $(a''^{(n-1)}, b''^{(n-1)})$ will be alike; and thus we can continue to trace backwards to the first equation, which answers our supposed conditions, viz., either that in which the first and last terms obtain contrary signs for the first time or the next succeeding one, selecting that of the two in which the last term is less than $\sqrt{C}$; this equation and all the succeeding necessarily produce periodical terms.

Corollary. In the extraction of the square root the equation is of the form

$$a'x^2 - a = 0, \qquad x = \frac{\sqrt{aa'}}{a'}.$$

In this case $aa' = C$, and therefore the least of the two numbers a, a' is necessarily less than $\sqrt{C}$; if a be the least, then x is the square root of a proper fraction, and the periods must commence from the first operation; but if a' be the least, the next transformed equation or second operation will produce a periodical term.

There are many other remarkably elegant properties and applications of continued fractions, but, being unconnected with the theory of equations, we must omit them on this occasion.

ON THE FORMATION OF PARTICULAR CLASSES OF EQUATIONS.

(105.) To form an equation of which the roots are all the natural numbers taken positively and negatively to infinity, and including zero.

Let us first form a function $F_n(x)$, of which the roots are $0, 1, 2, 3 \ldots n, -1, -2, -3, \ldots -n$, that is

$$F_n(x) = c.x(x-1)(x-2)\ldots(x-n)(x+1)(x+2)\ldots(x+n).$$

Hence

$$F_n(x+1) = c.(x+1).x(x-1)\ldots(x-n+1)(x+2)(x+3)\ldots(x+n+1)$$

therefore, $$F_n(x+1) = \frac{x+n+1}{x-n} . F_n(x).$$

Now in the limiting case, where n is infinite, let $\phi(x)$ be the function sought; and since $\frac{x+n+1}{x-n}$ then becomes -1, we have

$$\phi(x+1) = -\phi(x).$$

To solve this equation let $\phi(x) = C\varepsilon^{mx}$;

hence, $$C\varepsilon^m.\varepsilon^{mx} = -C.\varepsilon^{mx},$$

or $$\varepsilon^m = -1;$$

therefore $\quad m = \pm\pi\sqrt{-1}, \quad \pm 3\pi\sqrt{-1}, \quad \pm 5\pi\sqrt{-1}$, &c.

Hence the particular forms the sum of which gives the most general form of $\phi(x)$, so as to coincide with the preceding equation of differences, are the following:

$$\begin{array}{l} A \sin \pi x + B \cos \pi x, \\ A' \sin 3\pi x + B' \cos 3\pi x, \\ A'' \sin 5\pi x + B'' \cos 5\pi x, \\ \quad \&c. \qquad \&c. \end{array}$$

The effect of employing the equation of differences was to insure that the roots of these functions formed an arithmetical progression; but in the problem they must include zero; therefore the cosines must be rejected; also the sines of the odd multiples of πx vanish for fractional as well as integer values of x; the only function therefore which strictly agrees with the conditions of the question is $A.\sin(\pi x)$.

In a similar way we should find that the equation of which the roots are

$$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\ldots -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \text{ \&c. is } A\cos(\pi x).$$

Let it now be required to form an equation of which the roots may be

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3} \ldots\ldots \frac{1}{2^n}.$$

Let the sought function be represented by $\phi(x)$;

that is $\quad \phi(x) = (x-1)\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2^2}\right)\left(x-\frac{1}{2^3}\right)\ldots\ldots\left(x-\frac{1}{2^n}\right).$

Substitute $2x$ for x, and separate the numerical factors of x.

Hence $\dfrac{\phi(2x)}{2^{n+1}} = \left(x-\frac{1}{2}\right)\left(x-\frac{1}{2^2}\right)\ldots\ldots\left(x-\frac{1}{2^n}\right).\left(x-\frac{1}{2^{n+1}}\right),$

which, being compared with the preceding function, furnishes the equation

$$(x-1).\frac{\phi(2x)}{2^{n+1}} = \left(x-\frac{1}{2^{n+1}}\right)\phi(x).$$

Now $\phi(x)$ is of the form

$$x^{n+1} - a_1 x^n + a_2 x^{n-1} - a_3 x^{n-2} + \text{\&c.};$$

therefore

$$\left(x-\frac{1}{2^{n+1}}\right).\phi(x) = x^{n+2} - \left(a_1 + \frac{1}{2^{n+1}}\right).x^{n+1} + \left(a_2 + \frac{a_1}{2^{n+1}}\right)x^n - \left(a_3 + \frac{a_2}{2^{n+1}}\right)x^{n-1}$$
$$\text{\&c.}$$

Again, $\quad \dfrac{\phi(2x)}{2^{n+1}} = x^{n+1} - \dfrac{a_1}{2}x^n + \dfrac{a_2}{2^2}x^{n-1} - \dfrac{a_3}{2^3}.x^{n-2};$

therefore

$$(x-1).\frac{\phi(2x)}{2^{n+1}} = x^{n+2} - \left(\frac{a_1}{2}+1\right).x^{n+1} + \left(\frac{a_2}{2^2}+\frac{a_1}{2}\right)x^n - \left(\frac{a_3}{2^3}+\frac{a_2}{2^2}\right)x^{n-1} \text{ \&c.}$$

Compare the general coefficients in both these products, and we find

$$a_p+\frac{a_{p-1}}{2^{n+1}}=\frac{a_p}{2^p}+\frac{a_{p-1}}{2^{p-i}},$$

or, $$a_p(2^p-1)=a_{p-1}\left(2-\frac{1}{2^{n-p+1}}\right);$$

thus, $$a_1=2-\frac{1}{2^n},$$

$$a_2=\frac{2-\frac{1}{2^{n-1}}}{2^2-1}\cdot a_1,$$

$$a_3=\frac{2-\frac{1}{2^{n-2}}}{2^3-1}\cdot a_2,$$

&c. &c.

To form an equation of which the roots are $-\alpha_1$, $-(\alpha_1+\alpha_2)$ $-(\alpha_1+\alpha_2+\alpha_3)\ldots\ldots-(\alpha_1+\alpha_2+\alpha_3+\ldots\ldots+\alpha_n)$.

In this case we are to form the product

$$(x+\alpha_1)(x+\alpha_1+\alpha_2)(x+\alpha_1+\alpha_2+\alpha_3)\ldots.(x+\alpha_1+\alpha_2+\alpha_3+\ldots.+\alpha_n).$$

Represent this product by $x^n+A_1x^{n-1}+A_2x^{n-2}+\ldots+A_mx^{n-m}+\ldots$ $+A_{n-1}x+A_n$, the general coefficient A_m being the sums of products, each of which contain m factors.

For A_1 it is easily seen that its value is

$$n\alpha_1+(n-1)\alpha_2+(n-2)\alpha_3+\ldots 2\alpha_{n-1}+\alpha_n.$$

Again, A_2 consists of products such as $\alpha_1\alpha_2$ $\alpha_1\alpha_3$ $\alpha_2\alpha_3$, &c., and pure powers, as α_1^2, α_2^2, &c., the general form of the first class of terms is $\alpha_p\alpha_q$; and we now proceed to find its coefficient or the number of times this combination occurs, which number may be denoted by $(\alpha_p\alpha_q)$; and supposing p less than q, no factor preceding $x+\alpha_1+\alpha_2+\ldots.\alpha_p$ will be concerned in forming the combination in question, and in the factor itself and the succeeding ones, the terms preceding α_p may also be neglected.

The factor commencing from the above, arranged horizontally, will form this diagram:

$$\begin{array}{l}
x+\alpha_1+\ldots.+\alpha_p\\
x+\alpha_1+\ldots.+\alpha_p+\alpha_{p+1}\\
x+\alpha_1+\ldots\ldots\alpha_p+\alpha_{p+1}+\alpha_{p+2}\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
x+\alpha_1+\ldots\ldots\alpha_p+\alpha_{p+1}+\alpha_{p+2}+\ldots.+\alpha_q\cdot *\\
x+\alpha_1+\ldots\ldots\alpha_p+\ldots\ldots\ldots\ldots\ldots+\alpha_q+\alpha_{q+1}\\
x+\alpha_1+\ldots\ldots\alpha_p+\ldots\ldots\ldots\ldots\ldots\ldots\alpha_q+\alpha_{q+1}+\alpha_{q+2}\\
\text{\&c.} \qquad\qquad\qquad\qquad\qquad\qquad \text{\&c.}
\end{array}$$

Now if α_{q+1} were placed where the asterisk stands, the combinations of α_p with α_q and α_{q+1} would be alike $\therefore(\alpha_p\alpha_q)-(\alpha_p\alpha_{q+1})=$ the number of combinations of one term at the asterisk with the terms in the ver-

tical column of a_p, except *that* α_p, which is in the same horizontal line with the asterisk; it is therefore the number of terms minus one in that column which (since $p-1$ factors precede the first above written) will be $n-p$,

Therefore Δ denoting the finite difference when q increases by unity, we have

$$\Delta(a_p a_q) = -(n-p);$$

therefore $\quad (\alpha_p a_q) = (n-p)(c-q) \quad c$ being independent of q.

Suppose $q=n$, $(a_p a_n)$ will be the number of terms minus one in the column of a_p, since a_n enters only once; that is, $(a_p a_n) = n-p$, therefore $c-n=1$, or $c=n+1$, which gives

$$(\alpha_p a_q) = (n-p)(n-q+1)$$

As for the coefficients of the powers, as a_p^2, denoting such by a similar notation (α^2_p), they will not be affected by the supposition that $\alpha_1=0$ $a_2=0 \ldots . a_{p-1}=0 \quad a_{p+1}=0 \ldots . a_n=0$; they are therefore the same as in $(1+a_p)^{n-p+1}$, that is $\dfrac{(n-p)(n-p+1)}{1.2}$, which is half of the formula obtained by putting $q=p$.

Hence,

$$A_2 = \frac{n.(n-1)}{1.2}.a_1^2 + \frac{(n-1)(n-2)}{1.2}.a_2^2 + \frac{(n-2)(n-3)}{1.2}.a_3^2 + \ldots .$$

$$\begin{aligned}(n-1)(n-1)a_1a_2 + (n-1)(n-2)a_1a_3 + (n-1)(n-3)a_1a_4 + \ldots .\\ +(n-2)(n-2)\alpha_2a_3 + (n-2)(n-3)a_2\alpha_4 + \ldots .\\ +(n-3)(n-3)a_3\alpha_4 + \ldots .\end{aligned}$$

In like manner we may classify the terms of which A_3 is composed into terms of the forms $\alpha_p a_q a_r$, $a^2_p a_q$, a^3_p, respectively, p, q, r, being arranged according to magnitude, their coefficients may be represented as before by the same letters in brackets.

Every combination of $a_p a_q$ may be combined with a_r, except such as are formed from the a_p and α_q, which are in the same horizontal line with it: if these are erased, the number n is reduced to $n-1$, and the combinations of $a_p a_q$ are then, by what has been already shown, only $(n-p-1)(n-q)$ in number; therefore the excess of the number of the combinations of α_r with $a_p \alpha_q$ above that of α_{r+1} is $(n-p-1)$ $(n-q)$, or taking the finite difference in reference to r,

$$\Delta(a_p\alpha_q a_r) = -(n-p-1)(n-q);$$

thence $\quad (a_p a_q a_r) = (n-p-1)(n-q)(c-r);$

and putting $r=n$, we find, as before, $c=n+1$;

therefore $\quad (a_p a_q \alpha_r) = (n-p-1)(n-q)(n-r+1);$

and generally, if $s>r \quad r>q \quad q>p$, &c., then by the same process

$$(a_s a_r \alpha_q a_p \ldots .) = (n-s+1)(n-r)(n-q-1)(n-p-2) \ldots .$$

Again, if we erase the a_p, which is on the same horizontal line with a_r, the number of combinations of the remaining terms α_p, in number $n-p$, are $\dfrac{(n-p)(n-p-1)}{1.2}$; and since the number of terms in the

vertical line where α_q stands is $n-q+1$, it follows that

$$(\alpha^2_p\alpha_q)=\frac{(n-p)(n-p-1)}{1\quad 2}.(n-q+1);$$

and generally, $(\alpha_s^{s'}\alpha_r^{r'}\alpha_q^{q'}\dots)=\frac{(n-s+1)(n-s)\dots(s' \text{ times})}{1.2\dots s'}.$

$$\frac{(n-r)(n-r-1)\dots(r' \text{ times})}{1.2\dots r'}\cdot\frac{(n-q-1)(n-q-2)\dots q' \text{ times}}{1.2\dots q'}\dots$$

Lastly, $(\alpha_p{}^3)$ is the same as if all the terms α_1, α_2, &c. were zero, except α_p, and is therefore

$$\frac{(n-p+1)(n-p)(n-p-1)}{1.2.3}$$

More generally $(\alpha_p{}^{p'})\qquad=\frac{(n-p+1)(n-p)\dots(p' \text{ times})}{1.2\dots p'}$

We have thus investigated the coefficients of every combination which enter the whole product, and it may be remarked that the coefficients of the combinations of consecutive terms are pure powers; thus,

$$(\alpha_1\alpha_2)=(n-1)^2\quad \alpha_2\alpha_3=(n-2)^2 \text{ \&c. } \alpha_1\alpha_2\alpha_3=(n-2)^3.$$

This example is extracted from a memoir of the Author's, published in the "Transactions of the Royal Society for 1837," and is of essential use in the investigation of general formulæ for the change of the independent variable in the "Differential Calculus."

(106.) Having now given most of the known properties of algebraic equations containing a single unknown quantity, accompanied by illustrative applications, the length to which we have been conducted in these researches precludes us from noticing some other subjects of increasing interest; we refer in particular to the "Theory of Elimination;" we shall therefore conclude with some remarks on the roots of equations of which the dimensions are infinite; a subject which has various important applications, but is surrounded with difficulties, and has hitherto received but little attention.

Such an equation may either have no root real or imaginary, in which case the series corresponding to its left member is not capable of reversion, or it may have a finite or infinite number of roots, the reverse series giving the least when real.

In regarding the proposed as the *limit of an equation of finite dimensions*, it is convenient to choose the latter, so that the coefficients may be multiplied by factorials of successive orders, because on the hypothesis of any integer value for the order n, the general infinite series will be always reduced to a finite equation of n dimensions: this is the method which I purpose to exemplify as appearing to be most simple and general.

Given $\qquad 1+x+x^2+x^3+\text{\&c. } ad\ inf. =0.$

The derivative series which seems best to assume in this case is

$$1+nz+n(n-1).z^2+(n)(n-1)(n-2)z^3+\text{\&c.}=0,$$

which is a finite equation when n is any positive integer and is reducible to the proposed by making $z=\frac{x}{n}$, and then supposing n infinite.

To discuss the derivative equation, let $z=\frac{1}{y}$, when it becomes

$$y^n+ny^{n-1}+n(n-1).y^{n-2}+n(n-1)(n-2)y^{n-3}+\&c. =0,$$

represent the left member by u and its derived by u', and we find

$$u=y^n+u'=0.$$

Now if $u=0$ admits of real roots, let them be represented by α, β, &c., α being the greatest, β the next, &c. regarding negative quantities as succeeding positive in magnitude. If α be put for y in u' it must give a positive result; and since it renders $u=0$, therefore it must render y^n negative, which is impossible when n is even; there can be then no real root.

But as there must be a real root when n is odd, and that a renders y^n negative, therefore α and consequently any other real roots are negative, and would also render y^n negative; but β must render u' negative, and therefore y^n positive: hence we see that in this case y has but one real root.

Now if we put 0 for y in u, it reduces itself to its last term, which is positive; and if we put $-\sqrt{n}$ for the same, it becomes negative; therefore y is between 0 and $-\sqrt{n}$, z between $-n^{-\frac{1}{2}}$ and $-\infty$, and x between $-n^{\frac{1}{2}}$ and $-\infty$; therefore when n is infinitely great this root becomes $-\infty$. This equation therefore admits of no root, the series ceasing to give any expression in the latter case. We see in fact that the series is equivalent to $\frac{1}{1-x}$, which cannot vanish except when $x=\infty$, and then the series cannot be said to be represented by that fraction.

To find the roots of the equation $x-\frac{x^3}{2.3}+\frac{x^5}{2.3.4.5}-\&c. =0$, we form in like manner the derivative equation

$$nz-\frac{n(n-1)(n-2)}{2.3}.z^3+\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}.z^5-\&c.=0,$$

which is terminating when n is a positive integer; then put $x=nz$, and finally make n infinite.

Now the latter equation is the same as

$$\{1+z\sqrt{-1}\}^n-\{1-z\sqrt{-1}\}^n=0, \quad \text{or} \quad \left\{\frac{1+z\sqrt{-1}}{1-z\sqrt{-1}}\right\}^n=1;$$

and putting $z=\tan(\theta)$, we find $\left\{\frac{\cos\theta+\sqrt{-1}\sin\theta}{\cos\theta-\sqrt{-1}\sin\theta}\right\}^n=1$

or $\cos 2n\theta+\sqrt{-1}\sin 2n\theta=1$, whence $2n\theta=0,\ 2\pi,\ 4\pi,\ 6\pi..(2n-2)\pi$

and $z=0$, $\pm\tan\frac{\pi}{n}$, $\pm\tan\frac{2\pi}{n}$, &c.; making now n infinite, we find $x=0$, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, &c., π denoting the number 3,14159.....

Let the proposed equation be

$$1-x+\frac{x^2}{1.2}-\frac{x^3}{1.2.3}+\text{\&c.}=0;$$

and take for the derivative

$$1-nz+\frac{n(n-1)}{1.2}.z^2-\frac{n(n-1)(n-2)}{1.2.3}z^3\ \text{\&c.}=0,$$

or

$$(1-z)^n=0,$$

which has only the root $z=1$, therefore $x=\infty$, or there is no finite quantity which will satisfy the proposed.

Our last example is one, which though differently treated, has already attracted the attention of mathematicians, not so much with a view to its applications in the highér parts of analysis, and its remarkable connexion with important physical problems, but as a test for merely algebraical theorems.

Let $1-\frac{x}{1^2}+\frac{x^2}{1^2.2^2}-\frac{x^3}{1^2.2^2.3^2}+\text{\&c.}=0;$

the series which I take for its derivative is

$$1-\frac{n}{1}.\frac{n+1}{1}.z+\frac{n.(n-1)}{1.2}.\frac{(n+1)(n+2)}{1.2}.z^2-\text{\&c.}=0;$$

all the roots of which we have already proved to be real and positive, and to lie between 0 and 1; and I have shown in my "Treatise on Electricity" that the difference of two successive roots is of the order $\frac{1}{n}$; hence the limits of all the values of x may be easily found by the substitution of the natural numbers for x, and by observing the alternations of signs which result.

Thus, when $x<1$, all the terms within brackets of the following series are obviously positive, viz.:

$$(1-x)+\frac{x^2}{2^2}\left(1-\frac{x}{3^2}\right)+\frac{x^4}{2^2.3^2.4^2}\left(1-\frac{x}{5^2}\right)\ \text{\&c.}$$

there exists therefore no root less than unity, or equal to it.

When $x=2$ the sum of the 3 first terms vanishes, and the series becomes $-\frac{2}{3^2}\left(1-\frac{2}{4^2}\right)-\frac{2^3}{3^2.4^2.5^2}\left(1-\frac{2}{6^2}\right)-$ &c. which is negative, consequently there must be a real root between 1 and 2; and it is easy to be shown that there is but one. When $x=3$ the sum of the first five terms is negative, and all the succeeding terms taken in pairs give necessarily negative results, and *à fortiori* we find the same for quantities between 2 and 3; there is therefore no root in this

interval, nor as far as $x=4, 5, 6, 7, 8$; but the results in all these cases are very small and rapidly diminishing. If we denote the proposed function in this case by u and by u_1, u_2, &c., those from which it is successively derived, each vanishing when $x=0$, also by u', u'', &c., its own successive derived functions, it possesses the following property, which expresses the relation between its derived and primitive functions $u_m-(-x)^m u''^{(m)}=0$; thus, if $m=1$, we have $u_1+xu'=0$; and since the substitution of the roots of $u=0$ in u' being all real would produce a series of alternations in the signs, the same would occur to u_1 in the contrary way, x being always positive. The definite integrals of the function u when multiplied by other functions, possess very remarkable properties, intimately allied with the interior arrangement of latent electricity in bodies; but one object in selecting it here was to exemplify the method we have suggested above for examining transcendents not algebraically expressible, and to attract the reader's attention to a function as remarkable in nature as in analysis. The whole subject of equations infinite in their dimensions would require more space and consideration than can be here conveniently permitted.

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